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G. H. Gooderham.

LIFE ASSURANCE PRIMER

A TEXT-BOOK

DEALING WITH THE PRACTICE AND MATHEMATICS OF LIFE ASSURANCE, FOR ADVANCED SCHOOLS, COLLEGES AND UNIVERSITIES

BY

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PREFACE TO SECOND EDITION.

I am much gratified to find that the reception of this volume has been so cordial, and that there is an urgent and immediate demand for a new edition. There were only one or two misprints in the first issue, and these have been corrected. A new chapter, dealing with subjects of practical interest, has been added. The forms of statements supplied by Companies annually show where information concerning the various companies may be obtained; and the difficulties in dealing with comparative figures, which are fully discussed, will explain to students and others some of the dangers which lurk in the misapplication of statistical methods. Changes and additions have been made to bring the book up to date and many questions have been enlarged and illustrated to make the meaning more clear, with the result that thirty-two pages have been added to the new edition.

The only criticism of any consequence which has been made against the "Primer" is that it is not sufficiently elementary. But the book was not written for children; nor was it prepared for those who, with merely a general education, could read it through at a sitting, and at once understand it all. It was meant to be a study for men, willing and able to devote time to a careful reading, and *anxious to obtain true knowledge of the subject* to the extent of taking pains to digest it, and even to read other collateral publications. Life Assurance is a source of unbounded blessing to men; even more so to women and children. But knowledge of the subject does not enter our minds intuitively; it

can only be acquired by diligence, and it would be well if this social obligation were made part of the general school training.

One of the commonest errors is to assume that life assurance is an investment, and that premiums paid for protection are like deposits in a savings bank. It is difficult to convince many men, even intelligent men, who have paid for life assurance for ten years and who have lived, that their premiums should not *all* be held by the Company increased by interest. Yet if this were so the heirs of those who died could only get the amount of their own premiums, with interest, less expenses; while we know that the full amount assured is paid even if death take place only a few days after payment of the first premium. This is the fundamental difference between a savings bank and a life assurance company. If such general principles were known more widely, many would avoid those disappointed expectations which have caused so much heartburning.

Wide-spread interest has been aroused by the various investigations of Life Assurance Companies. These have had the effect of directing the public gaze to an important subject; and, although much of the information given in sensational daily papers is highly colored, and often incorrect, still the permanent effect on the minds of the people will be for good. We cannot talk on any subject without gradually imbibing more and more of the truth about it, and therefore the freer the discussion the better. If this book enables a few people to correct wrong impressions which have crept into their minds, the author will feel that his work is not in vain.

HENRY MOIR.

New York, 1*st January,* 1907.

INTRODUCTION.

There has been a tendency within the last few years to extend University Education in the direction of life assurance, treated sometimes from an economic standpoint and sometimes mathematically; and the want of a Text-Book dealing with elementary principles has been greatly felt. It is presumed that those reading the technical portions of the following work already possess a slight knowledge of algebra, rather better than that usually afforded by the High School course. The subjects have been explained as far as possible in a simple way, so that the reader who may not have this algebraic knowledge may yet be able to grasp the general outlines, even though he may pass over many of the mathematical demonstrations.

For those who propose to complete their actuarial studies, many books are available, in particular the Text-Book of the Institute of Actuaries, which is recognized as the authoritative work on Actuarial Science. That work, however, is elaborate, and is designed for advanced students; whereas this Primer, as its name implies, has been prepared for those who may wish to obtain a proper understanding of the principles, but who are not yet prepared to undertake the more advanced study.

The notation used in the Institute Text-Book was adopted by the International Congress of Actuaries, after giving careful consideration to other systems. That notation is therefore used herein so far as it is necessary, and a brief explanation is incorporated in the appendix. There is also included in the appendix a sufficient number of tables to afford the means of illustrating

the subjects discussed, and give students some practical knowledge of the questions with which they may deal.

One of the aims of the work has been to include such information on actuarial subjects as every official of a life assurance company should possess. The Actuary and his assistants should have a full grasp of the details, with technical knowledge to solve the more intricate problems; but general principles should be mastered by the other officials, so that all departments of the work may be conducted in harmony.

LIFE ASSURANCE PRIMER.

CHAPTER I.

PRINCIPLES OF LIFE ASSURANCE, AND OF LIFE ASSURANCE COMPANIES.

Mutual Protection. The general principle of Insurance consists in the combination of many persons to protect each one against unforeseen loss. The principle is perhaps most easily understood in connection with such contracts as those of Fire Insurance. The risk of fire is slight, and the loss when a fire does occur is frequently heavy. If this loss were borne by one person alone it might mean ruin; but if he had combined with some thousands of other persons, all holding property, and all insuring against fire, then the loss when spread over the whole number of persons would be the merest trifle to each. It is assumed that the danger of fire in each case is the same. When one class of property is more liable to fire than another, a suitable variation is made in the premium. The same principle runs through life assurance, though in a more complex form. If all persons who combine for life assurance were of the same age, the position would be simple, and much the same as fire insurance. But although an individual might die at any moment, young men live on the average for many more years than old men, and to secure equity the premium payments are graded according to the age at entry, that is the age when a policy is first taken. Then too, different people want different kinds of assurance, as is explained in the next chapter, and each kind of policy has its own scale of rates, so that there is a good deal of complexity even in the simpler phases of life assurance.

Means of Saving. While the principle of mutual protection is at the foundation of life assurance, one very forcible argument in its favor at the present day is that it affords one of the best means of saving money. Human nature is weak and good resolutions to set apart a portion of one's income are seldom carried out. Even if a considerable saving has been effected, the temptation of luxury, or a speculative investment, may nullify the self-sacrifice of years if the money is lying there ready at hand. The premiums under a life assurance policy come round with unfailing regularity, and when once they are paid the consideration that the money is set aside for good reduces the chance of its being spent.

Insurance or Assurance. The words "insurance" and "assurance" are generally used with exactly the same significance. Probably the most useful distinction which has been drawn between the two is that the word "assurance" should apply to protection against an event which is certain to happen at some time, the date of happening being unknown. Of such nature is Life Assurance. And that the word "insurance" should be applied to an event which may or may not happen. Of such nature are Fire and Marine Insurance, where the property may be insured for an indefinite time without there being any necessity for claiming compensation for loss under the contract. This distinction is, however, purely arbitrary, and is only occasionally used, although suggested by Charles Babbage about eighty years ago. "Assurance" is used here because this word has more of an international character, and we should do all we can to promote the universal use of the same word. The initial letter "A" is used in actuarial works with a definite significance, now understood over the whole world.

Origin and Growth. Life Assurance has grown up gradually, and it is difficult to trace its exact origin, which may have been in the old funeral and burial societies, as these are of very ancient date. They provided that all members of the Society should have a satisfactory burial; and, in addition, they fre-

quently made payments to widows and orphans. Then, of somewhat later date, there were fraternities and guilds. These bodies, on the death of any deserving member in poor circumstances, would make a call upon the other members for contributions. Such contributions were rather of the nature of charitable donations than of benefits to be obtained by right.

It is sometimes said that the practice of modern Life Assurance by regular companies grew out of Marine Insurance. In very early days insurance was effected on ships and cargo when a voyage was in contemplation. As the loss was frequently serious if anything befell the captain of the vessel, it became usual to effect a policy on his life for the voyage, the usual charge for the protection being five per cent. This premuim had no scientific foundation; indeed, no system of Life Assurance which was really sound, and in accord with present-day principles, was adopted until the formation of the old "Equitable Society" in London in 1762. This society was, and still is, a mutual organization for the assurance of its members, and no agents have ever been employed. A level premium is charged and returns of surplus are made to policy-holders at periodic intervals.

Whatever the origin of Life Assurance, there is no doubt that it has now become a "necessity of modern society". It enables any man of ability, without capital, to capitalize his ability, and undertake obligations which he would not otherwise be justified in doing. He can provide that obligations will be met in event of his premature death. Many persons also view Life Assurance as the best means of saving; while even the wealthy appreciate its advantages to protect their estates from forced realization.

Scientific Basis. It has been shown by investigations, conducted on many occasions and in many different ways, that the chance of death at any age for men of average health is a very definite quantity. While it is quite impossible to say at what date any individual is likely to die, yet if a large group of persons all of the same age be under consideration, it can be foretold with wonderful accuracy that a certain number will probably die in the first year; a certain other number in the second year; a third group in the third year; and so on until the total

number passes out of existence at the extreme limit of old age. This regularity provides the scientific basis on which calculations are made. The soundness of the basis is sustained by the experience of a century and a half, and by the gigantic proportions to which Life Assurance has developed as one of the principal branches in Social Economy.

Forms of Assurance.

The different forms of Life Assurance in the United States are generally distinguished as follows:

(1) Assessment;
(2) Old Line, including Ordinary and Industrial;
(3) Stipulated Premium; and
(4) Fraternal.

It seems unfortunate, however, that these distinctions exist, and especially that the different forms of organizations should be controlled under different laws. The subject of Life Assurance seemed already to be sufficiently complicated by the diverse laws in different states; and the system adopted by the legislatures of distinguishing Life Assurance into several different kinds, and providing separate laws for each, has been the cause of further confusion.

ASSESSMENT ASSURANCE.

This form originated in the funeral societies, guilds, and friendly societies already mentioned; but its development was very slow until about thirty years ago, when it was adopted and forced to the front in America. It is therefore one of the oldest forms of mutual protection; at the same time, it is of such a simple nature that its characteristics can be easily explained. For these reasons it is discussed and disposed of at once—occasional references later will show more clearly the reasons for some statements now made.

Under the original forms of this class of assurance it was usual to charge an assessment of fixed amount at the end of each year, and pay the amount thus received to the heirs of those members of the society who might have died during the preceding year. The amount of the payment at death would therefore

vary according to the number of deaths, and it was usually necessary to have some strong force to bind the members together other than the mere assurance benefit. This cohesive force frequently consisted in benefits derived from membership of a guild. The next development was in the direction of fixing the payment at death, and letting the assessment on the surviving members vary according to the number of claims.

Assessment Illustration. In order to illustrate the working of such plans, let us consider the simplest case of a society with a large number of members all of one age and all young men. The annual assessment charged against each surviving member would at first be slight, because the deaths at young ages are comparatively few. But gradually the assessments would increase with the increasing ages of the members; and, at the same time, unless the membership were recruited annually, there would be a smaller number of persons remaining to pay these assessments. It would therefore be natural that at the end of each year, when the assessment levies were being made, a number of members would withdraw, refusing to pay their assessments, and this would increase still further the payments required from the diminished membership of such a society. In order to avoid this difficulty as far as possible, the practice arose of paying fixed amounts and charging assessments in advance, estimating at the beginning of the year the amount which would probably have to be paid by reason of deaths. Assessments paid in this way were therefore properly called "premiums".

Assessment Assurance Impractical. Under the fundamental principle of assessment assurance, as above explained, the premiums should increase with each year of age to meet the increasing death rate; but assessment companies, especially in recent years, have generally held out the hope to their members that the premium would remain at the original rate for an indefinite period, and possibly even for the whole of life. If the premiums were actually increased each year, and if all the members were intelligent enough to understand that they were paying for

protection from year to year only, the *theory* of this class of assurance would doubtless be satisfactory. But when premiums begin to increase members of such companies always do become dissatisfied and many of the healthy withdraw. The unhealthy members perforce remain with the company; and, by their increased death rate, make the losses of the company still more serious. The withdrawal of members and the increasing death rate therefore interact on each other. The increased death rate causes the withdrawal of members, and this withdrawal of members causes a still further increased death rate, so that the effect is cumulative.

There are many companies in America, and friendly and funeral societies in Great Britain, which operate on this plan with more or less modification. Scientifically its basis may be sound, but for the practical affairs of human life it has by experience been shown to be entirely unsatisfactory because of its lack of stability and permanence. This form of assurance cannot therefore in any sense be recommended; and this feeling has been so strong of recent years that laws have been passed in many states prohibiting the formation of new companies to operate on the assessment plan.

OLD LINE ASSURANCE.

While the original crude forms of Life Assurance in all countries were undoubtedly for temporary purposes only, or arranged upon the assessment plan, the first real development in the United States took place on safe and scientific principles in the form now known as "Old Line". But the rigid enforcement of valuation laws during the period of depression and weakness from 1870 to 1872 caused widespread liquidation amongst such companies, and so many of them became insolvent that a new order of things sprang up. To avoid the valuation laws, assessment companies and fraternal orders were formed one after another and grew to gigantic proportions in a marvellously short time. It is only in very recent years that this backward movement has been stopped, after many had learned a lesson by bitter experience.

Adequate Premiums Necessary. Old Line Life Assurance requires that the premiums charged for the protection shall be scientifically adequate to secure the benefits under the policy, regardless of the form such benefits may take. The assurance may run from year to year only, in which case the premiums must increase with the increasing age; but this plan is open to much the same objections as assessmentism. More frequently, however, the benefit is guaranteed for a number of years, or for life, and the premiums may then be paid by equal annual instalments over the term, or throughout life. Or, a fixed sum payable at death may be secured by single payment, or by annual payments for a temporary period, as will be afterwards fully described. The general principle is that a certain benefit is guaranteed and that an adequate premium is charged for the benefit. If the premiums prove to be more than adequate, an adjustment is generally made by means of a distribution of surplus at a subsequent date.

INDUSTRIAL ASSURANCE.

The benefit of Old Line Assurance is offered in small sums with premiums payable weekly or monthly under the title Industrial Assurance, an illogical name which arose from the fact that this form of assurance has been largely adopted by the industrial classes. In the United States there are nearly fifteen millions of such policies in force, and the sums assured aggregate about $2,000,000,000. A weekly premium of five cents at age 30 will provide a sum assured at death of about $67. Premiums are collected from door to door, and when claims arise they are generally paid within a day or two, because the companies are empowered to hand over the money to any relative of the deceased, or to any one incurring funeral expenses. These Industrial Companies are therefore performing a great work, not only because of the millions they pay annually to widows and orphans, but also because they are educating the masses in habits of thrift, and teaching them the benefits of sound Life Assurance. They are also rapidly extending their sphere and leading the better class workmen to take larger policies by annual premiums and thus secure fuller protection at smaller proportionate cost.

Industrial Expenses. Much criticism has been applied to Industrial Life Assurance because of the expense incurred. This expense, while it is to be deplored, is necessary to the kind of business. The cost of collecting five or ten cents a week is great in proportion to the amount collected; and the only way to make any great change in this respect seems to be to teach workingmen to set aside part of their weekly wage and pay a quarterly or an annual premium. Meantime the Industrial classes can frequently pay as much as fifty cents each week, but could not pay the equivalent $6.00 each quarter. The remedy lies in the education of the people, not in the destruction of a system which, in spite of its extravagance, is doing an immense amount of good.

STIPULATED PREMIUM COMPANIES.

Certain companies have been formed on the basis of charging the smallest possible premium adequate to the risk, but with what may be called a *safety clause* in their policies to prevent legal insolvency in times of depression. The safety clause gives them the right to charge any deficiency in their assets against the policy-holders themselves, and call upon the policy-holders to make good such deficiency by additional premium payments. If the additional premiums be not paid a similar result may be secured by reducing the sum assured. The main difference therefore between such companies and those of the regular "Old Line" form is that the sums assured in Stipulated Premium companies are not absolutely guaranteed for a fixed premium. A somewhat similar effect is sometimes obtained in an Old Line company where a portion of each annual premium may be treated as a loan secured against the policy, to be repaid from the surplus, or dividends, if these are sufficiently large. In Great Britain, in like manner, several companies issue policies at very low premium rates "reduced in anticipation of future bonuses"; they discount future surplus.

Under some of the special laws which have been formed for the regulation of Stipulated Premium companies, it is provided that they must charge a *premium at least equal to the rate of mortality for a single year*, with a minimum loading for expenses

and contingencies. Any company charging this rate of premium only would be on a par with an assessment company; but some of them charge premiums closely approaching the uniform life rate, and endeavor to furnish protection for life at a minimum premium. Such companies are permitted to issue policies guaranteed for the whole of life at an annual premium, and even to provide that payment of premiums may cease after a limited number of years if they carry as a liability what is known as the net value of such policies.

FRATERNAL ORDERS AND FRIENDLY SOCIETIES.

These two terms are nearly synonymous, the former name being that commonly used in the United States and the latter in Great Britain. While the bases may be alike, there is great difference in the methods of working, and especially in the growth which has taken place amongst those fraternal orders founded to provide Life Assurance benefits for their members. The Friendly Societies of Great Britain have generally some object in common quite apart from life assurance, such as Masonic interests, or common occupations; while on the contrary Life Assurance is the principal object of many large Fraternal Orders in the United States.

Definition and Principles. A Fraternal Beneficiary Association has been defined thus: "Any association, without capital stock, "organized and carried on solely for the mutual "benefit of its members and their beneficiaries and not for profit, "and having a lodge system with ritualistic form of work and "representative form of government, and which shall make pro- "vision for the payment of death benefits". The general principle underlying these bodies is therefore that they are organized and conducted solely for the benefit of their members on what may be called the co-operative plan. Each member then becomes an agent for the introduction of others. It is usual in the fraternal orders to have a ritual of a more or less elaborate nature, including special ceremonies for the initiation of members and secret signs; while social gatherings form an attractive feature. When

the order has been organized with Life Assurance as one of its main objects a medical examination is required. If an applicant be unhealthy and cannot pass the medical examination, it is a common custom to permit him to become a *social member* of the order on the payment of certain reduced fees.

Fraternal Development. These fraternal orders were almost unknown in the United States before the outbreak of the civil war; but their growth, like that of the industries, has been phenomenal in its rapidity and extent, so that at the present day there is a membership of about 4,500,000 persons, whose assurance benefits aggregate probably $5,000,000,000. There are also several orders in which membership is confined to women, and whose officers and trustees are all women. Originally these orders were founded on the assessment plan. A call was made on the surviving members, depending in amount on the number of deaths, and the sum thus subscribed was distributed amongst the beneficiaries, provided it did not exceed a certain fixed maximum for each death. This method, however, was not satisfactory, and now the death benefits are generally fixed in advance, and are met by means of entrance fees and fixed premium payments depending upon the age at entry. The organizers of such orders in most instances had no sound knowledge of the principles of Life Assurance. Frequently all entrants were charged the same scale of contribution, irrespective of their age. This was unjust to young members, who, after living to pay the claims by death of their seniors, often found their premium charges becoming burdensome with advancing years, with only a vague prospect of any claim being made good on their own death. The result has been that a large number of these societies ceased to exist through the increased mortality rates causing their healthy members to withdraw. The lifetime of such organizations is therefore short, seldom longer than twenty or thirty years. With their increasing age the questions of heavier mortality and expense become more and more urgent.

Fraternal Conventions. Representatives from the principal Fraternal Orders in the United States meet annually in convention, and these meetings have had great influence in fostering sound views. In 1896 a special committee reported that an adjustment of rates was becoming imperative in many societies, and recommended that "contributions shall be equitably proportioned to the hazard at risk". To the expert this statement is a mere platitude; but not so to many of the officers of Fraternal Orders who were accustomed to hear all sorts of fallacious arguments regarding enormous profits to be earned from lapses and other sources which do not exist in such orders. The dangerous condition of these bodies was exposed so clearly that many of them have been adopting measures to charge sufficiently large premiums, and place their assurance benefits on a sound and lasting basis. Some societies have adopted what is called the "step-rate", or yearly-renewable-term plan of assurance, with premiums increasing yearly; and others have attempted the better, if more difficult, task of changing to what would practically be ordinary whole life premiums.

Sound Principles. As the chance of dying increases with increasing age, it follows that unless premiums also increase a larger payment must be made in the early years than is immediately necessary for meeting the risk of death then incurred. This additional payment cannot be treated as profit, but must be carefully husbanded and accumulated at interest to provide the sum assured at a time when the same annual premium would otherwise be insufficient to meet the risk. This is fundamental in sound life assurance, although it has often been stated that the large accumulations of Old Line companies are unnecessary. As the best method of exposing error is to disseminate truth, little more need be said about fallacious and unsound methods—correct principles commend themselves when they are properly understood.

MUTUAL AND PROPRIETARY COMPANIES.

There are two classes of companies which offer Life Assurance on the Old Line plan:—

1. Those conducted and controlled by the policy-holders themselves for mutual assurance; and
2. Those conducted and controlled by shareholders, whose capital forms an additional security for protection of policy-holders.

Although the difference in the forms of constitution is considerable, yet, for practical results, there is really little to choose between the two classes of companies. In Proprietary Companies, as well as in Mutual Companies, the interests of policy-holders take first place. In most of them the vast bulk of the profits which are earned are distributed amongst the policy-holders themselves. Sometimes Proprietary Companies which distribute surplus amongst policy-holders are called "mixed" companies, a peculiar title, drawing a distinction which, at the present day, is scarcely necessary. In many cases the dividends to shareholders are limited to a definite percentage, so that when account is taken of the interest earned by the shareholders' funds, the balance taken from the profits of the company is the merest fraction of the entire surplus.

The principal difference therefore between the two consists in the method of electing directors. In Mutual Companies these are elected by the policy-holders, and in Proprietary Companies by the shareholders. In the United States the votes are controlled in Mutual Companies by proxies obtained from policy-holders, in Proprietary Companies by actual control of the stock. In practice, therefore, there is little difference even on this score, because the management of a Life Assurance company must necessarily be of a permanent character to ensure success. Frequent change in management generally points to disintegration. Accordingly, whether the directors and executive officers are appointed by shareholders, or by policy-holders, their offices are usually held more or less permanently—in other words, they are to a great extent elected by themselves.

CHAPTER II.

Different Kinds of Assurance, and How to Obtain a Policy.

Life Policies may generally be classified in the divisions:—

1. Whole Life;
2. Term;
3. Endowment; and
4. Special Contracts.

A fifth class, Annuities, will also be discussed, as they have a close connection with the subject.

WHOLE LIFE ASSURANCE.

Ordinary Life. Whole Life Policies are those wherein the sum assured is payable at death and at death only. They are essentially for the benefit of others: in their early years they generally afford protection for a wife or children, while in the case of prolonged life the sum assured is then available for legacies or bequests at a time when the children have attained independent positions for themselves. When premiums are payable throughout the entire lifetime the policy is usually called an "Ordinary Whole Life Assurance", or a "Whole Life Assurance by Continued Payments".

The following may be taken as a very simple form of such a policy:—

> In consideration of the application for this assurance, hereby made part of this contract, and the payment in advance of an annual premium of $250.00, and the payment of a like amount on the first day of January in each year during the continuance of this contract, the World Life Assurance Company agrees to pay the sum of $10,000 upon receiving satisfactory proofs of the death of John

Smith, residing at No. 100 First Street, New York City, to his executors, administrators, or assigns.

IN WITNESS WHEREOF this contract has been signed by the President and Secretary of the WORLD LIFE ASSURANCE COMPANY at New York, this first day of January, 1904.

This specimen contract is reduced to its simplest possible terms. It does not take into consideration several Privileges which are granted to the policy-holder; and, in like manner, ignores Conditions which are desirable for the protection of the company.

Privileges. The Privileges generally inserted may be briefly summarized as follows:—

1. A month is usually allowed for payment of premiums after the due date, the assurance protection being continued during these "days of grace".
2. On discontinuance of the policy, Surrender Values may be claimed by the policy-holder. A value may generally be taken in cash, or in the form of reduced sum assured on which no further premiums are payable, or by continuing the assurance at the full amount for a temporary period without payment of premiums. In the United States it is usual to include a table of such values in the policy. In Great Britain the same custom is being introduced, although heretofore the practice has been to state only a minimum cash value, such as 33 per cent or 40 per cent of the premiums paid.
3. Provision is made for granting loans to the policy-holder within the value of the policy at a moderate rate of interest.
4. Participation in the surplus funds of the company may be granted.
5. The privilege of changing the policy to one of another form is sometimes given, and it is usually stated that the policy shall be "incontestable" either from date of issue or after the lapse of one or two years.

Conditions. The Conditions which are more or less necessary from the company's standpoint are:—

1. That the age of the life assured be correctly stated;
2. That the information submitted with the application for assurance be truthful; and
3. That the life assured have no immediate intention of
   (a) Committing suicide;
   (b) Engaging in a hazardous occupation; or
   (c) Proceeding to an unhealthy climate.

The above privileges and conditions apply to all classes of policies, and not only to the Whole Life policy already mentioned.

Beneficiary and Insurable Interest. Another valuable privilege consists in the right to nominate a beneficiary to whom the sum assured shall be payable. In the foregoing specimen policy the sum assured is payable to "executors, administrators or assigns", but contracts are frequently drawn so as to provide that it shall be payable to wife, or to a child, as Beneficiary; and, when this is done, the policy is generally protected against the claims of any creditors of the life assured: the policy constitutes a trust for the purpose for which it was effected. On the question of relationship, it may be said in general that dependents have the right to effect assurances on the lives of those who support them. Thus a wife may take a policy on the life of her husband, a daughter on her father; but an uncle may not do so on a nephew unless for reasons other than mere relationship. In the case of policies drawn in favor of strangers, no vested interest is acquired unless there is what is known as an "insurable interest". Without such insurable interest the effect of nominating a stranger as beneficiary would be much the same as bequeathing the sum assured by will; the bequest would be revocable by the life assured at any time. If, however, there is a pecuniary interest held by one person in the life of another, the former may effect a policy on the life of the latter to protect himself against loss in event of death. Legal restrictions have very properly been placed upon effecting assurance on the lives of persons in whom there is no direct interest, because such

practice, which at one time was not uncommon, constitutes "a pernicious kind of gambling". When the assured pays the premiums, and reserves the right to change the beneficiary at pleasure (a common condition), the claims of creditors would probably prevail over those of a beneficiary, at least during the lifetime of the assured.

Liberal Interpretation. The tendency of recent years has been to observe much caution as to the character of persons admitted to the benefits of life assurance; and, when once they have been accepted, to interpret the policy contract with the greatest possible liberality. Particular care is observed as to the habits and social surroundings of applicants before the issue of any policy, because it has been found by long experience that loose, drunken, or immoral lives are inevitably subject to heavy mortality rates.

Limited Payment Policies. To many thoughtful persons the continued payment of premiums in the later years of life is distasteful. They do not know whether their income in old age may be sufficient to continue the annual premiums. The desire in this direction has been the cause of introducing Whole Life policies by Limited Payments, one of the most popular forms, and deserving all its popularity. Under such a policy the sum assured is payable at death, but premiums are only charged for a fixed number of years, after which the policy becomes paid-up for life. The life policy by twenty payments is probably the favorite of this class; but assurances are also written by thirty payments, twenty-five payments, fifteen payments and ten payments. Occasionally even a policy is taken by single premium, but this form of contract is of advantage only to a man of means. To such a person it forms a good investment for part of his capital, while providing assurance protection from the date of payment of the premium.

TERM ASSURANCE.

Sometimes protection is required for a temporary period or for temporary purposes only. In such cases the face value of the policy is payable only in the event of death taking place dur-

ing the term fixed, and premiums are generally charged annually during such term. For business purposes a policy is occasionally desired for a period as short as a single year; while sometimes policies are issued on what is known as the Yearly-Renewable-Term Plan, under which each year's premium is in theory just sufficient to meet the average cost for the current year, so that the premiums increase with the increasing age of the policy-holder. In fraternal circles this policy is often called the "step rate". This form of policy is open to the same objection as that formerly mentioned in connection with assessment assurance, although not to the same extent. The policy-holder has the right to renew the assurance from year to year at certain fixed rates of premium, without any medical re-examination and whether he be healthy or unhealthy at the time of renewal. One of the objections to this form of policy is that it is exceedingly difficult to prove to any dissatisfied person who has paid premiums for ten or twenty years that he has already obtained value in protection for the premiums that he has paid, and that no part of his premiums can be refunded. Frequently this renewable term plan is applied to assurance at fixed rates for longer periods, five, ten, or twenty years. Taking by way of illustration a Ten Year Renewable Term policy, the premium remains at the same annual rate during the first ten years after entry. At the end of ten years, however, the premium then due is based upon the attained age, ten years greater than the original age at entry; the premium therefore increases each time the policy is renewed, that is, every ten years. Term Assurance of this kind has never found favor in Great Britain, but it has been used in America to a considerable extent. The twenty-year term form, with the option to convert to Whole Life or Endowment Assurance at any time within the first fifteen years at the premium for the attained age, has been written to a limited extent in Britain by one or two companies. The term policies in Great Britain are usually for periods of 1, 3 or 5 years only, in connection with financial or business transactions.

Renewable Term Plan.

In the case of a man who desires the largest possible protection for the premium he can afford to pay, and who can use all

his available savings in business, this form of policy is sometimes appreciated. As a general rule, however, it cannot be commended, because it contains no element for the accumulation of personal savings, and the premiums often become unbearably high at a time when protection is still required.

ENDOWMENT ASSURANCE.

The policies so far discussed provide sums assured payable only in the event of death; but one form of policy contract, known as the Endowment Assurance, provides for the sum assured being paid at a fixed date if the policy-holder be then living, or at his death if this should happen before the date of maturity has been reached. A policy of this kind appeals to the personal desire of many people, because it provides a means of saving money which may afterwards be enjoyed by those who have accumulated it. When the sum assured is payable at death only, it is necessarily paid to others; but in the case of an Endowment Assurance the sum assured not only protects others in the first instance, but may revert to the life assured himself on survival. This form of policy has attained great popularity during the past twenty years; and it is gradually taking the prominent place in Life Assurance formerly occupied by the Ordinary Whole Life policy. Premiums are most frequently paid during the entire continuance of the policy contract, but sometimes these policies are effected on the basis that the sum assured shall be payable after a longer term, such as thirty years, and that premiums be limited to ten or fifteen annual payments. An Endowment Assurance maturing after a fixed number of years is occasionally taken by single premium.

The most usual period for such policies in America is twenty years. In Great Britain the tendency is to select a longer term. The shorter period Endowment Assurances combine immediate protection with the saving of money; while the longer period policies combine rather the same immediate protection, with provision for old age. For the latter purpose, policies are frequently written as payable at death or on attaining a certain age such as fifty-five, sixty, or sixty-five. Sometimes even policies have been written as "Life Rate Endowments", with the sum as-

sured payable at age eighty or at previous death. This last mentioned form is, however, little more than a Whole Life Assurance, and has not attained the same popularity as those which mature at younger ages.

Pure Endowment. Care should be taken to distinguish the above contract from the Pure Endowment, which provides the principal sum on surviving a fixed period, but in event of earlier death no payment is due. In such case the premium is of course smaller than for the Endowment Assurance: it depends upon the likelihood of survival of the fixed period.

SPECIAL CONTRACTS.

These may be divided in two general classes, the *first* consisting of the combination of two or more of the forms already mentioned, or the combination of one of such forms with some savings bank principle; and the *second* class relating to policies involving more than one life. Of the first class particular meniton may be made of

(1) Instalment Policies;
(2) Policies with Increasing Premiums;
(3) Policies with Decreasing Premiums;
(4) Children's Endowments;
(5) Gold Bonds, or Debenture Bonds.

Under the second class a very large variety of contracts might be discussed, but attention will be confined to four of these only, namely:—

(1) Continuous Instalment Policies;
(2) Joint Life Assurance;
(3) Last Survivor Assurance; and
(4) Survivorship Assurance (one life against another).

Instalment Policies. Instalment Policies may be written on any of the usual plans of assurance. The peculiar feature is that the sum assured, instead of being payable in one sum at death or on maturity, is payable in annual instalments after death

or maturity. For example, instead of the company being liable to pay $10,000 in cash under a Whole Life policy at death, the stipulation may be that $1,000 shall be payable at the beginning of each year for ten years after the death of the assured, or that $500 be payable each year for twenty years. The advantage of a contract of this kind is that the beneficiary receives an income limited to a certain number of years, instead of a fixed sum in cash which might be dissipated or lost through unwise investment. Such policies are of course obtained at smaller cost in proportion to their face value than those payable in one sum.

Increasing Premiums. The most common form of Increasing Premium Policy, apart from the renewable term, consists in the combination of temporary assurance with life or limited payment life forms. For example, a policy may be written at a low rate of premium for the first five years under a temporary assurance plan, while the policy-holder may have the right to renew at a higher premium after the first five years expire; this higher premium being of the nature of continued payment life, limited payment life, or endowment assurance. A policy of this kind is principally of advantage to a man whose income is likely to increase rapidly, or whose capital is valuable to him in business during the immediate future. The yearly renewable term contract formerly mentioned is of course an example of an Increasing Premium Policy.

Decreasing Premiums. Decreasing Premium Policies are not popular. For all practical purposes they consist of a series of limited payment contracts combined together. For example, if the initial premium be reduced at the end of five years to three-fourths; at the end of ten years to one-half; at the end of fifteen years to one-quarter; and terminate altogether at the end of twenty years, the same result can be obtained by adding together the premiums required for twenty payment life, fifteen payment life, ten payment life, and five payment life policies, then dividing the sum by four.

Children's Policies. Under Children's Endowments it is frequently provided that in the event of the death of the child all premiums paid will be returned in full, but that the principal sum of the endowment shall only be payable in event of the child attaining a fixed age. A policy of this form therefore contains very little of the element of life assurance, but is simply a means of providing a sum in cash for educational or business purposes when the child attains a specified age. Sometimes it is provided that no return of premiums will be made, but this stipulation is far from popular, although it admits of a smaller premium being charged.

Another form of such an Endowment provides that, in event of the death of the purchaser of the Endowment before all premiums are paid, the premiums shall thereafter cease, and the Endowment become paid up in full for the benefit of the child on attaining the stipulated age. The purchaser is frequently the father or some other near relative, and if such a condition be made it is of course necessary that he be in good health. The premiums are higher than those where no such condition applies. Several companies in Great Britain have introduced a policy for children under which premiums would be repaid in event of the child dying before attaining age 21; and, on attainment of that age, the assurance would become an Ordinary Life, Limited Payment Life, or Endowment Assurance contract, without medical examination. The rate of premium is a very low one, frequently less than $10 per $1,000, even for an assurance with premiums ceasing absolutely at the age of 50. Such policies, however, furnish protection for a remote period, and for this reason are a means of saving money for the benefit of a child rather than procuring assurance in the usual sense.

Gold Bonds. The Gold Bond or Debenture Bond is of a somewhat similar nature in its construction to a policy by instalments. It is generally provided that a Gold Bond or Debenture Bond shall be given to the beneficiary instead of cash when the policy matures, so as to provide a safe investment for the policy moneys. The principal sum assured is accordingly

retained by the company, and it is agreed that interest will be paid semi-annually for a number of years, after which the principal itself will fall due. Any rate of interest may be guaranteed on this Bond, the premium payable under the assurance contract being adjusted in such a way as to provide for interest at the rate stipulated. For example, in the case of a 5 per cent Gold Bond the premium calculations may be made on the assumption that 3 per cent interest will be earned by the company on the sum assured in its possession; while the additional 2 per cent, making 5 per cent in all, must be provided from the extra premium charged for this class of contract. For a still larger premium 6 per cent interest could be guaranteed, and so forth.

Continuous Instalment Policies. This name is given to a form of policy which combines the advantages of a reversionary annuity with life assurance. The usual form of such contracts provides that the face value of the policy shall be payable in twenty annual instalments *guaranteed;* and, in consideration of an additional premium payment, that the instalments be continued after the twenty years so long as some specified beneficiary may survive. This latter portion is therefore of the nature of a reversionary annuity. The additional payment for this further provision is generally small because of the contingencies involved, viz.:—the beneficiary has to survive the life assured, and not only so, but has to live more than twenty years after his death before any benefit is derived from this stipulation. In event of the beneficiary dying in the lifetime of the life assured, the premium is generally reduced to the regular rate necessary to provide the sum assured after death in twenty annual instalments. The same feature may be applied to endowment assurance policies with a guaranteed income to the life assured himself, or to the last survivor of himself and a beneficiary.

Joint Life Assurance. Under this form of policy the sum assured is payable on the occurrence of the first death amongst two or more persons. It is sometimes an advantageous form of policy for partners in business, because on the death of one partner the business might be seriously crippled

through the withdrawal of capital; while a Joint Life Policy (often called a Partnership Policy) would provide the capital which might have to be withdrawn, and permit of the continuance of the business with the same efficiency as before. Sometimes also such policies are desired by husband and wife who enjoy a joint income. Three or even more lives may of course be introduced under one policy contract, but it is only in exceptional circumstances that such policies are desirable from the standpoint of the lives assured. When more than two persons are involved in partnership, the likelihood of change in business relationships is largely increased. Any change of this kind introduces complications as to the assurance policy, and an intricate division of interests, which can best be avoided by taking out separate policies on individual lives. Moreover, as the lives are increased in number there is a greater likelihood of one or other of them being ineligible for assurance.

Joint policies may be effected on any of the forms already discussed:—Whole Life by Continued Payments or Limited Payments; Term Assurance; Endowment Assurance, etc.

Last Survivor Assurance. While under the Joint Life policy the sum assured is payable at the *first* death, under the Last Survivor policy it is payable at the *last* death. Such policies are seldom required, although sometimes when two persons have an income which will be continued to the survivor, and they desire to borrow money on their joint interest, a policy of this nature may enable them to effect their purpose on reasonable terms.

It will be at once understood that a joint assurance on two lives, and a last survivor assurance on the same lives, provide together exactly the same protection as two separate policies on the individuals, because the joint policy provides the sum assured on the first death, and the last survivor policy on the second.

Contingent or Survivorship. Policies assuring one life against another are common in Great Britain, but are not so frequently issued in the United States. They provide for the sum assured being payable at the death of some

person, *only* if another specified person be living at the date of such death. Such policies will be understood more readily if the circumstances under which they are generally issued be explained.

It is common in the will of a wealthy man to provide that the entire income from his property be paid to his widow, and that the property be divided on her death amongst certain heirs or legatees who may then be living. In such circumstances it is evident that the share of the property would be lost by any heir or legatee who might die during the lifetime of the widow. The cheapest form of protecting this share from absolute loss is the Survivorship Assurance, providing the sum assured at his death in event of its occurring in the lifetime of the widow. Assurance companies occasionally grant loans secured by contingent interests in estates to be divided at some future time, called Reversions, and any such loan should be protected by a survivorship policy.

Further complications are frequently introduced under policies of this general class when three or more lives are involved in one contingency; but it is not necessary to do more than mention the fact of such complications, which cause great variety in the contracts of this character.

General Remarks. Much has been said in explaining special contracts, and the reader might therefore think that these contracts were the most important in Life Assurance. This is not by any means the case. A very large percentage, probably three-quarters, of the entire business in America is written on three policy forms, namely: (1) Continued Payment Life; (2) Twenty Payment Life; and (3) Twenty Year Endowment Assurance. These are the important classes. The explanations regarding them may seem brief and inadequate in view of their importance, but the policies are easily understood, and for that reason commend themselves to the public.

The Limited Payment Policy, with the right to share in accumulated surplus at the end of the premium term, is frequently effected with the same object in view as an Endowment Assur-

ance, and a great many of these policies are surrendered when they become paid-up. They are even discontinued when the policy-holder may still need assurance protection: he frequently takes the cash value and accumulated surplus of the one policy and immediately effects another. This is a course which cannot be commended, because he has to pay a much higher rate of premium at his attained age; and, moreover, by continuing the older policy he would frequently obtain further participation in surplus without payment of any more premiums, a privilege which he forfeits on surrendering the policy. The temptation of the cash value available at that time should therefore be resisted.

ANNUITIES.

The most common form of Annuity is that granted in consideration of a cash payment, the company agreeing to pay a fixed sum annually, semi-annually, or quarterly, so long as the annuitant may survive. This contract therefore assures a fixed income to the purchaser, whether life be brief or prolonged. The annuity generally ceases, and all benefit is lost, from the date of death. It will be seen therefore that this contract is really the converse of Life Assurance by Continued Payments. Under the Life Assurance policy the price consists of an annual payment during life, and the benefit is the sum assured, due when premium payments cease. Under the annuity contract the price consists of a cash payment due at once; and the benefit is an annual income during life. The close relationship between the assurance and the annuity contracts is shown technically in chapter VIII.

Guaranteed Minimum Payments. Occasionally it is stipulated that a fixed *minimum number* of annuity payments shall be made. For example, in consideration of the purchase price of $1,000 an annuity of $100 might be granted payable during the lifetime of the person nominated; and, whether such person live or die, providing that at least ten annual payments should be made.

Deferred Annuity. As a rule, the first annuity payment is due three months, six months, or a year after the date of purchase. Sometimes, however, a deferred annuity is granted, subject to the provision that annuity payments shall not commence until the purchaser attain a certain age, or survive a fixed period of years. This form of annuity provides a *future income*, and it may either be purchased for a single premium payment, or for annual premiums during the period of deferment, or by limited premium payments. Most frequently, in event of death taking place, no refund is made of the purchase price or premiums; but sometimes under a deferred annuity it is stipulated that in event of death before the first annuity payment has fallen due, the premium or premiums which have been paid shall be refunded in full to the heirs of the purchaser.

Last Survivor Annuity. It is not uncommon for husband and wife, or for two sisters, to purchase an annuity payable to them jointly while they are both alive, and continuing at the full rate to the survivor. By this means an income is provided so long as the survivor of the two can possibly require it. The same principle may, of course, be extended to three or more lives, but the circumstances are rare when such annuities are desirable, while for two lives it is a common form of contract.

Reversionary or Survivorship Annuities. The annuities so far discussed are of a nature almost exactly the opposite of Life Assurance. Under a Reversionary Annuity, however, the conditions are much nearer those of assurance contracts, because the annuity is entered upon by the annuitant or beneficiary after the death of the purchaser, who may be called the life assured. The most common case is that of a husband who desires to provide an income to his wife from the date of his own death. During his lifetime he pays an annual premium, and from the time of his death the company guarantees to pay a fixed income to his widow. As a rule, the entire benefit is lost if his wife should predecease him, but sometimes the stipulation is made that the premium will, in such circumstances, be-

refunded. The Continuous Instalment policy, as previously mentioned, is just a reversionary annuity, with the addition of a guaranteed minimum number of annuity payments.

STANDARD POLICY FORMS.

Forms Prescribed. The recent laws enacted in New York provide that after 1st January, 1907, all policies issued by New York companies within the State "shall be in the forms "hereby prescribed and not otherwise, save as hereinafter pro-"vided". The forms prescribed are:—

(1) An Ordinary Life Policy;
(2) A Limited Payment Life Policy;
(3) An Endowment Policy;
(4) A Term Policy; and
(5) Such other forms as may be adopted by the Superintendent of Insurance after certain formalities.

The wording of these forms has been fixed by the Legislature, subject to a power of modification, within the limits of the law, by the Superintendent of Insurance prior to 1st October, 1906.

Standard Privileges. The Standard Forms give to policy-holders many rights which were formerly *privileges* offered by the companies as inducements to effect policies. The most important examples are:—

(1) The right to change the beneficiary; (2) The right to have a grace period for payment of premiums; (3) The provision that the policy shall be incontestable not later than two years after it has been issued; (4) The right to obtain loans; and (5) The right to take settlement of the sum assured in other ways than in cash, such as annual instalments or an annuity for life. Companies are not bound to issue such forms in other States or countries.

Doubtful Benefit. It is very doubtful if the use of standard policies will prove either advantageous or popular. There are so many varieties of policies, and so many different wants to be supplied, that the forms cannot be reduced to any moderate

number without companies having to refuse to grant requests which are not unreasonable. By prescribing forms in any rigid manner, legitimate competition of the best kind is destroyed, and progress towards liberality ceases. The public loses the benefit. This has been recognized by other legislators who have considered the subject since the laws were passed in New York. The report of the commission appointed by the Governor of Massachusetts, in accordance with a resolution of the State Legislature, shows careful thought and conservatism, and states: "It is our conviction that any standard forms for life insurance "policies are undesirable. Comparison of the policies issued to-"day with those issued twenty years ago teaches that competi-"tion among the companies has brought about steady progress "in liberality to the insured. . . . Instead of attempting to "standardize the policies, we think that certain provisions should "be standardized, and the companies left free to adopt policies "containing any further provisions not inconsistent with those "required by law."

Procedure in Obtaining a Policy.

Application or Proposal. When a policy of life assurance is wanted, it is first of all necessary to fill out an application form. In Great Britain it is usually called a "proposal", the person desiring the policy being there called the "proposer". This application, or proposal form, contains a schedule of questions eliciting Name, Age, Residence, Occupation (past and present), Date and Place of Birth, amount and plan of assurance desired, whether married or single, whether contemplating any change in residence or occupation, name of Beneficiary and relationship to applicant, and calling for details of assurance effected in other companies, or of any previous rejection by another company of a similar application. It generally closes with a warranty that all the statements are true; that the applicant has no intention of proceeding to an unhealthy climate or undertaking any hazardous occupation; and that if any statement be untrue, the policy issued on the basis of such application

shall in consequence be void. But under the recent laws it is provided that all statements in the application shall be representations only, and not warranties. Fraud in obtaining the policy must therefore be proved in order to avoid the policy obligations.

Medical Report. After completing the form above mentioned the applicant has to appear before some medical man appointed by the company to conduct an examination on its behalf and report upon the risk. Questions are again submitted regarding Name, Age, etc., together with particulars of past illnesses, the use of alcoholics or narcotics, and the causes of death of parents and other relatives, with the present state of their health if living. These replies are then warranted as true and signed in the presence of the Doctor, the replies and signature forming one of the means of identification. This is a needful precaution, because the applicant is in many cases unknown to the Doctor; and, without this signature, a third party might be substituted for examination. The Doctor then makes a thorough physical examination of the applicant, writes out the details of this, and is occasionally asked to state whether he considers the subject in good average health. This report is forwarded to the Home Office of the company, and forms one of the principal records regarding the life proposed for assurance.

Inspection Report. When the application for assurance is of small amount, such as $1,000 or $2,000, it is usual in America to issue a policy on the basis of the application and medical report alone, without further particulars; but for all larger policies it is the practice to have an Inspection Report from some local source, quite independent of the agent who procured the application, or the applicant himself. This inspection, which is conducted on principles similar to those of a private detective agency, provides information particularly about the habits and mode of life of the applicant, shows whether he is sober and temperate, and reports also upon his financial circumstances.

Private Friends' and Agency Reports. In Great Britain the Inspection Report is not much used, although when there is any reason for suspicion a report from a commercial agency is sometimes obtained. In place of the inspection, it is the custom there to take reports from two private friends, dealing particularly with the habits of the applicant, and also a report from the agent in the case, dealing with his financial circumstances, occupation, etc. These reports, however, are all obtained from persons who are more or less interested in the applicant and in the issue of a policy. The private friends are usually companions, and they naturally place a very lenient interpretation upon the words "sober and temperate", These reports are therefore of little practical value in determining whether an applicant for assurance should be accepted or not.

Issue of Policy. When all these papers have been received at the Home Office, they are carefully examined to ascertain that the various statements are in harmony with one another. If all the information is satisfactory and the application is for an average amount, it is usually passed by the Medical Director without further formality. Applications for large amounts, those which are not approved or are doubtful, and those requiring extra premiums, frequently form the subject of special consultation between the Medical Director and one of the Executive Officers, or are discussed in committee. If the application be approved, the policy is written and forwarded to the agent who hands it to the applicant in exchange for the first premium. In Great Britain, legal decisions have rendered it preferable to write only an official receipt for the first premium when the proposal for assurance has been accepted. This official receipt is used by the agent for collection of the first premium, and the policy contract itself is not drawn until *after* the first premium has been paid.

CHAPTER III.

MORTALITY.

Tables of Mortality, which form the scientific basis for life assurance, were very little used until towards the end of the 18th century, after the Northampton Tables had been published. Other tables were in existence nearly 100 years before, one of the first being that compiled from the statistics of the population of the town of Breslau in Silesia by the British Astronomer-Royal Halley, and published in 1692. Halley at the same time advocated correct principles for annuities and assurances, which were not adopted until nearly 70 years later.

There are two principal sources for obtaining such information as is necessary for compiling a good mortality table, namely:

(1) Population statistics, including registers of births and deaths; and

(2) Life Assurance Statistics.

Other sources are occasionally used, such as particulars of peerage families, which have been carefully recorded and published for many generations; widows' and pension funds; employees in large corporations; army and navy statistics, etc., etc.; but the two above mentioned are sufficient for our present consideration, as they bring under review all the salient features of mortality.

STATISTICS OF POPULATION.

In order to form tables correctly from population records, it is necessary to have an enumeration of the people with their ages, together with a record of the deaths which take place and the ages at death. Even with such statistics, the results are influenced by movements in population in the nature of emigration

or immigration during the period over which the observations extend. Frequently, also, when an enumeration of population is taken, the ages are misstated or given approximately. Discriminating judgment is essential in dealing with mortality statistics—otherwise erroneous conclusions may be reached.

Northampton Table, 1783. The Northampton Table may be taken as an interesting example of erroneous construction, because Dr. Price, who published the table in 1783, ignored several important factors affecting the mortality. The statistics on which he based his results consisted of a record of the deaths in two parishes in the town of Northampton. No enumeration of the population was taken, but the tables were formed from the deaths, with ages at death. Dr. Price had also a record of the baptisms which had taken place in the community, and he found that the number of deaths exceeded the baptisms in the period under observation (1735-1780). He therefore *assumed* that the additional deaths were caused by immigration into Northampton at the age of 20. As a matter of fact, however, the additional deaths were largely the result of the baptisms being less numerous than the births. There were many Baptists in the town, and the names of their children did not appear on the records of christenings. The baptismal records were on an entirely different basis from the death records, and they should not have been used together. The assumption above mentioned was therefore inaccurate, and moreover the mortality of the two parishes from which the figures were taken was higher than the average of other towns; so that the Northampton Table was in several respects unreliable. It was, however, extensively used for many years; and, as it showed excessive mortality, large profits were earned by life assurance companies which adopted its figures. On the contrary, annuity companies suffered severely because their annuitants lived longer than the table indicated. It is a strange and unaccountable fact that this Northampton Table is still used in the United States for certain legal purposes, and in some courts appears to be the only recognized table of mortality.

Carlisle Table, 1815. On the other hand, the Carlisle Table may be quoted as one which was prepared on scientific principles, and from satisfactory statistics. This table was published in 1815 by Joshua Milne. It was constructed from a census of the population of two parishes in Carlisle in 1780, and the deaths in the same parishes from 1779 to 1787 inclusive. It is usually supposed that a second census was taken in 1787; but the figures themselves indicate that this is doubtful. Anyhow, it was found that an increase (treated as being exactly 1,000) in the population had taken place, and allowance had to be made for this increase as affecting the deaths also, so that the ratio of the deaths to the numbers living could be satisfactorily obtained. There was a larger proportion of female than of male lives included in the statistics, and the result was to show light mortality at the older ages. The table has been very extensively used, and even to the present day is considered of great value, especially in the calculation of survivorship benefits. For general life assurance purposes, however, it has been superseded by tables formed from mortality amongst assured lives.

Census Tables. Mortality tables from the general population of England have been formed on five occasions, the latest having been published in 1897, dealing with the period from 1881 to 1890. The mortality of the entire population was given and compared with that in various occupations. A most useful series of tables was submitted, with details of mortality in one hundred different occupations, showing not only the mortality rates in those occupations, but giving also the causes of death.

Healthy English Table, 1861. One of the population tables has received more prominence than others, namely: The Healthy English Table. It was formed by Dr. Farr from the census returns of 1851 and the records of the births and deaths, from 1848 to 1853 inclusive, in 63 of the healthiest registration districts of England and Wales. In all of these districts the mortality of the general population did not exceed the rate of 17 annual deaths to 1,000 living; and at the census of 1851 the

total population in these selected districts was nearly one million persons, of whom about 493,000 were males and 503,000 females. The mortality of the sexes was investigated separately. This table (male section) was selected by a committee of the Actuarial Society of America for comparison with the Specialized Mortality afterwards referred to, with slight modifications at ages under 21 and at ages from 51 to 60, and with a special adjustment for the reduced mortality arising from medical selection during the first five years of assurance.

Life Assurance Statistics. The records of life assurance companies are almost free from the errors and misstatements to which population statistics are subject. The ages of those who take out policies are correctly taken, and a complete record is kept from which can be obtained the number under observation at any age, and the number of deaths which take place amongst them. The first table formed from the experience of a life assurance company was prepared by Mr. Arthur Morgan, Actuary of the old Equitable Society, and published in 1834. The records of any individual company are not, however, so valuable as records formed from a group, because individual companies have frequently peculiar conditions affecting their mortality, while a group is more likely to represent a fair average, such as may be equalled by any well managed corporation.

Actuaries' or Combined Experience Table. Accordingly, in 1843, there was published the experience of 17 life assurance companies, now known in America as the "Actuaries'" or the "Combined Experience" Table. The statistics from which that experience was compiled embraced nearly 84,000 policies running from 1762 to 1833, of which nearly 14,000 terminated by death. The average duration of all the policies was less than 8½ years. The mortality amongst females was taken out separately and it was found that between ages 20 and 50 the mortality was considerably *heavier* than amongst males, but the reverse was the case above age 50.

American Experience Table, 1868. The American Experience Table of Mortality, now recognized as the standard table in the United States, was formed by Sheppard Homans and was first published in a schedule attached to an Act passed by the legislature of the State of New York on 6th May, 1868. The author never gave full particulars of the data employed. It is generally supposed that he used the mortality statistics deduced from the experience of the Mutual Life Insurance Company of New York as his basis, but these figures were inadequate at the older ages and accordingly he must have arbitrarily adjusted the table. It has been very much used, and has grown in popularity because it is found to represent faithfully the American mortality amongst assured lives after the first effects of selection have disappeared. It is now generally prescribed in the state laws as the standard for valuation purposes. Moreover, the table has been successfully graduated by Makeham's law of mortality, so that the calculation of complex benefits is thereby made comparatively simple.

H^M Table, 1869. The Institute of Actuaries' Tables were formed from statistics contributed by 20 British Companies, and the most important section of the experience is that known as the Healthy Males or H^M. Particulars of about 180,000 policies were submitted and analyzed, but of these some were on the lives of persons who carried two or more policies, and about 11% were on female lives. By excluding duplicate policies and female lives the number was reduced to little over 140,000 healthy male lives, and these form the basis of the H^M Table. The ages at entry were kept separate so that the effects of the medical examination could be investigated, and this was afterwards done by Dr. Sprague; but, in the published H^M Table, all those of the same attained age were dealt with together, irrespective of the age at entry. It was found that the low mortality of recent entrants had practically disappeared after five years, and accordingly another table was formed, called the $H^{M(5)}$, from which the first five years from entry (more properly 4½ years) were excluded from the statistics. The mortality rates are higher than

those of the H^M Table. A separate table was formed dealing with healthy female lives, the H^F Table.

The H^M Table has been used very freely in Great Britain, but has never found favor in the United States. In Canada, however, it forms the legal standard at the present time. The table starts at age 10 with a radix of 10,000, and very complete monetary values have been published.

British Assured Lives. The most important investigations of recent years are those conducted by the British Actuaries covering the period from 1863 to 1893. There are three particular features of that experience which had never before received so much attention and care, namely:—the tracing of the mortality (1) according to the duration of the policy, and (2) according to the kind of policy taken; and (3) the publication of full tables of Withdrawals. All of these are important influences, and as regards the mortality in policy classes, it was clearly proved that Endowment policies, and those which call for heavy premiums in proportion to the risk, were subject to lighter mortality rates than Whole Life policies and other forms where the premium rate is light as compared with the risk incurred. The reason for this is that those who take Limited Payment and Endowment Assurance policies are generally well-to-do people in their own sphere, thrifty and prudent, who look to the future and prefer a little self-sacrifice now if it will provide greater comfort hereafter. The prudence they show in their life assurance is an index to their whole mode of life, and the discretion they exercise is rewarded by length of days. The most important tables are those based upon Ordinary Whole Life policies With Profits, and in that class more than 550,000 lives were included. The effects of selection in that class of policy were also carefully analyzed for each year during the first ten years of duration, and full monetary tables have been published.

The following table shows the classes of policies which were separately investigated, and gives also the numbers of lives included in the aggregate observations:—

BRITISH ASSURED LIVES, 1863–93.

Male Lives:	No. of Lives.	Years of Risk.
(1) Whole Life Assurances, With Profits..	551,838	7,056,863
(2) Whole Life Assurances, Without Profits	56,807	602,591
(3) Endowment Assurances, With & Without Profits..................	132,043	897,673
(4) Whole Life, Limited Premiums.......	36,839	410,251
(5) Whole Life, Ascending Scale..........	23,280	207,709
(6) Joint Life Assurances................	9,195	90,171
(7) Contingent Survivorship Assurances...	3,482	Over 15,500
(8) Temporary Assurances...............	11,603	Over 36,000
Female Lives:		
(1) Whole Life Assurances, With Profits..	42,293	507,042
(2) Whole Life Assurances, Without Profits	11,050	112,010
(3) Joint Life Assurances.................	7,222	77,078

The Female Lives in the other classes are so few as to be of no value statistically.

Specialized Mortality Investigation. One very important investigation recently completed was that made by the Actuarial Society of America into special classes of assured lives. In all there were 98 separate classes of risks, covering a wide range of material, and the particulars of about 2½ millions of lives were furnished by the American life assurance companies. The classes of risks may be generally described as follows:—

(1) *Policies for large amounts;* one class over $20,000.

(2) *Policies granted on terms other than applied for,* (2 classes).

(3) *Nationality;* divided into 4 classes.

(4) *Occupation;* divided into 35 classes and covering army, navy, and marine service; the more important hazardous trades; liquor dealing; and railway service.

(5) *Personal Disability;* covering 32 classes, including

past history of diseases such as gout, blood-spitting, etc., unusual weights and unusual heights.

(6) *Family history unsatisfactory;* covering 2 classes dealing respectively with cancer and insanity.

(7) *Place of Residence;* 22 classes, each relating to a different county in the United States.

The mortality experience in these different classes was compared with a table which was adopted as representing standard mortality amongst healthy lives, with allowance for the effects of medical selection during the first five years from the date of issue. Great care must be exercised in making use of the results of this investigation, and discriminating judgment is necessary because of two important considerations which an inexperienced reference to the published tables might not at first reveal. The first and most important is that the lives assured in these special classes were all *accepted* by assurance companies; and presumably, therefore, they represented the very best material from these special classes. One prominent example may be given by way of illustration, namely, the class relating to personal history where the applicant "has had blood-spitting". The rate of mortality shown in that class is only 108 per cent of the tabular rate adopted as representing normal mortality. This cannot be considered as the true mortality amongst people generally who have suffered from blood-spitting; but it does represent the mortality which was experienced by assurance companies *on lives accepted* who had disclosed this feature in their past history. It may be taken as a certainty, however, that any person giving a history of blood-spitting within a period very shortly before the date of taking a policy would be absolutely rejected; and moreover, even in the case of a history of blood-spitting some years before the date of application, the case would be declined unless the physique of the applicant in other respects were almost perfect. If any applicant showed an under-average physique, and gave at the same time a history of blood-spitting, he would almost to a certainty be declined by any assurance company. The statistics do not always indicate the *average* mortality amongst persons

coming within any one of the specialized classes; but only the mortality experienced by companies on lives accepted by them—ignoring entirely rejected lives of the same class. Analogous remarks apply to many other cases; and, as already stated, the results must be used with great care and discrimination.

The second consideration is not of so much importance—it is the question as to whether the standard table has been accurately chosen as representing the average mortality amongst assured lives in America. If the standard show high mortality, then all the results appear more favorable than they ought; while if the standard be low then the opposite effect is obtained.

Mortality amongst Annuitants. Many other important tables of mortality have been formed, but there is no need to enter into minute details of these, as such details do not involve particular principles. Three different tables have been constructed from the experience of annuitants, more or less directly connected with the British Government schemes. The first two of these tables, published in 1829 and 1863 respectively, included some special lives carefully selected by speculators who purchased annuities and drew the proceeds for their own advantage, a practice which was afterwards discontinued by law in Great Britain. They also included certain old tontine funds formed in the 17th and 18th centuries. But the last of such tables, published in 1883, gave statistics of Government Annuitants only. The rates of mortality shown by the three sets of tables do not differ to any material extent, and they all prove that female annuitants are much healthier than male. The same feature is still more marked in the latest publication relating to the mortality amongst annuitants, namely: the British Offices Life Annuity Experience, published in 1903. *For example*, the expectation of life at age 50 of a male annuitant is 20.7 years, while the corresponding expectation of a female annuitant is almost 23 years. This experience embraced a period from 1863 to 1893, and included particulars of about 10,000 male lives and 25,000 female lives.

Male and Female Mortality. This condition as regards male and female mortality has been confirmed from many other sources. Amongst annuitants, however, even at the younger ages the mortality of females is less than that of males; and a careful distinction must therefore be drawn regarding the nature of the contract entered into. The reason for the different condition at younger ages probably is that annuities are generally purchased on the lives of spinsters and widows in good circumstances; while in life assurance transactions the same conditions do not so frequently apply.

With and Without Profits. The mortality amongst lives assured who take policies with the right to participate in surplus is noticeably lighter in Great Britain than that amongst those who take non-participating policies. The reason probably is that non-participating policies are generally effected on the lives of borrowers and for financial reasons; and, as already stated, debtors and borrowers are not on the average so healthy as thrifty people. There is reason to believe, however, that the same condition does not hold in America. It is thought rather that the opposite effect is experienced, viz.:—that non-participating policy-holders are subject to lighter mortality rates than those who take policies with the right to participation. The reason is that non-participating policies in America are taken with a purpose in view entirely different from that above outlined; in many cases this form is taken as the result of extreme conservatism.

SELECTION.

One important question in connection with the mortality of life assurance companies is the effect of what is called Selection. Before a policy is issued on the life of any person he has to undergo a medical examination and other restrictive tests designed to set aside all who are below a certain standard of vitality. The result is that persons who obtain policies of assurance are subject to lower mortality rates than the general population at the same age, and particularly is this noticeable in the

period immediately after the policies are taken. At that time there are none but healthy lives; of course even those are subject to accident and to what might be called accidental diseases, such as fevers, pneumonia, etc. By the recent O[M] experience the death rate per thousand amongst selected lives at age 40 is less than 5. In the course of a year or two, however, illnesses of a more permanent nature make their appearance amongst policy-holders: heart, lung, and brain diseases affect the lives to a greater or less degree, and indeed the mortality gradually approximates to the same rates as affect the classes of persons from amongst whom the selection had been made. At the same age of 40, amongst persons who have been assured for 5 or more years (i. e.:—who effected policies at 35 or a younger age and were 40 at date of investigation) the death rate is about 10 per thousand, or more than double the rate applicable to selected lives of the same age.

Right to Withdraw. Another feature which has received a great deal of attention as affecting the rate of mortality is the right always enjoyed by policy-holders of withdrawing and lapsing their policies, either by taking a surrender value in cash or otherwise. The opinion was formerly held quite generally that the healthier policy-holders discontinued their contracts, while those who were unhealthy and could not obtain assurance protection elsewhere, continued their policies year after year. It was therefore thought that the effect of lapses would be to increase the average mortality amongst those retaining their policies, as the tendency would be to lower the average vitality by the withdrawal of an undue proportion of healthy lives. Of recent years it has been pointed out that policy-holders who discontinue their contracts do so more frequently because of financial embarrassment than because they are healthy and do not require the protection. The protection offered by life assurance is more thoroughly appreciated after it is possessed than before; and healthy, prosperous people value the benefits more than the improvident. It follows, therefore, that a great many lapses take place because the policy-holder has fallen into irregular habits,

and it is well known that the rate of mortality affecting such people is much heavier than that which applies in the case of thrifty and prosperous men. Lapses of this nature tend to improve the mortality amongst those remaining, because they reduce the proportion which the unhealthy bear to the total number.

Discrimination by Policy-holders. As is indicated above, there is an element of discrimination on the part of policy-holders themselves according to the class of policy they may take. Those who take policies which provide large assurance protection at small rates are subject to heavier mortality than those who take policies of investment forms which require larger premiums, although each class is subjected to the same initial tests by the assurance companies. The circumstances and the basis of argument are exactly the same as those referred to in the preceding paragraph. Prosperous and thrifty people, who are looking to the future, prefer investment forms of policies which provide not only immediate protection, but at the same time build up a capital for themselves in later life. The man who is struggling along from year to year and living close up to his income may feel the necessity for assurance protection for the benefit of his family or of his business, but he strives to obtain this protection at the cheapest possible rate; and, as already indicated, men of this temperament are subject to heavier mortality rates.

Doctrine of Average. It is sometimes said that if a company were to issue policies for $1,000 each on the life of every person passing along a certain street on a given day, without medical examination or any other test, and at the ordinary premiums for the respective ages, and if the policies were duly maintained, the result would be a most profitable one for the company. This remark is probably quite correct; but it is not correct to pass from this statement, as is frequently done, and say that the companies could afford to take all who apply without selection. If any company were to do this, the unhealthy would apply for large amounts, because it is easy to convince an

unhealthy man that life assurance is desirable; while on the other hand the healthiest would refuse to be classed amongst them, and would rather take the risk upon themselves. In like manner, it would not be correct to accept every person passing along a certain street *for such amounts as they might themselves choose*, because in that case the healthy would take small amounts and the unhealthy large. The average therefore would not be maintained, because there would be a greater preponderance of business on unhealthy persons.

No single company can ever afford to dispense with careful selection, although it is possible that at some time in the dim future, when social science has developed much more fully than at present, the companies may unanimously agree (or legislation may require) that all who apply should be assured for an amount determined according to the assessed value of the life. Something in this direction might take place if it were possible to assess the monetary value of a person's life in a manner similar to the assessing of the value of property for fire insurance; and it is the only direction in which it would seem to be possible to conduct life assurance without medical examination in its complete sense.

Without Medical Examination. Various plans of life assurance without medical examination have been adopted and are more or less popular in Great Britain, but restrictive measures other than examinations are in vogue to secure that the experience under such policies will be of an average character. Mention may be made of the compulsory assurance of employees in the service of large corporations where every person must be assured, the premiums being often payable wholly or in part by the employer. In such case, the average is maintained because the assurance is obligatory on all. Or again, as the effects of medical selection are supposed to be of little consequence five years after policies have been taken out, (not that this supposition applies to individual cases in any way, but only *on the average*), one or two companies offer policies, subject to participation in the surplus of their own class, and with the

proviso that, during the first three or five years, only a portion of the face value would be payable in event of death. Another plan which has been found very successful is to issue only what is known as a Double Endowment Policy. Under such a contract if the life assured were to survive a fixed period, then the amount payable at that time as an Endowment would be double the amount which would have been paid in event of death during the period. In this case, also, such policy-holders participate in the surplus of their own class, and the premiums are sufficiently large to ensure that there will be a good surplus for distribution.

These plans are all said to be successful, but they could not under existing laws be practised throughout the United States, because in several states life assurance companies are expressly forbidden from issuing policies unless a medical examination of the life shall have been made by a duly qualified examiner.

GRADUATION.

When the rates of mortality have been ascertained from the data of assurance companies or from population statistics, it is usually found that they fluctuate from year to year and do not run smoothly. For example, in an individual case the rate of mortality at age 30 might be found lower than that at age 29, although other experiences would indicate that the contrary *should* be the case, while at age 31 again the rate might be abnormally high. Such accidental irregularities are to be expected, more particularly if the numbers under observation are small; and it is therefore necessary, in order to obtain a mortality table of practical utility, to submit these statistics to a process known as "Graduation". Graduation aims at removing irregularities which may be accidental, without disturbing what may be peculiar features of the statistics investigated. The adjusted results should be compared with the ungraduated figures. The tests of a good adjustment are stated to be:—

(1) General regularity in the mortality rates;

(2) Close agreement in the total number of deaths in graduated and ungraduated tables; and

(3) The frequency in change from positive to negative deviations, which indicate that the original figures have been closely followed.

The simplest methods of graduation consist merely in smoothing over the irregularities by averaging the results at two or three ages; and the process has been amplified from this, through the graphic method, by which the mortality is represented by a diagram, to the assumption of a law of mortality which has been found to adhere very closely to the facts in several large investigations, and which provides great facility for the solution of complex problems. It is, however, outside the scope of an elementary work to enter with any minuteness into this subject, which is one of considerable intricacy.

CHAPTER IV.

INTEREST AND INVESTMENTS—GENERAL SUBJECT.

Definition. Interest is the compensation paid by a borrower for the use of money. The actual form the money may take is quite immaterial. Americans naturally think of dollars; British students call to mind pounds sterling; Japanese deal with yen; anything of value which may be made reproductive can form the basis for interest calculations, and therefore reference will generally be made to 100 or to 1 or to 5% without specifying either pounds or dollars.

Rate of Interest. Interest is usually computed on the basis of each 100 lent, and it is payable at the end of each year, or half or quarter year. The *rate per cent per annum* is the form of measurement most familiar to the mind. In practice the rate of Interest varies according to

(1) Supply and demand affecting money at the time;
(2) Risk incurred by the lender of losing his money; and
(3) Ease with which the lender may recover it on short notice.

These three are the principal factors; but there are many minor considerations causing the Rate of Interest to fluctuate widely over short intervals; and also, with more gradual change, over longer periods.

Simple and Compound. The subject of interest is generally divided into two sections, Simple and Compound. If *Simple* interest is not paid when it falls due, but increases the indebtedness, this non-payment does not entitle the lender to interest on the overdue amount. Accordingly the simple interest

at 4% on 100 for 2 years would be 8. But *Compound* interest is different in that the interest, when due and unpaid, becomes part of the principal, and earns more interest. Thus the compound interest on 100 for the same time, namely for 2 years at 4%, would consist of 4 for the first year, and 4% on 104 for the second year, that is 4.16, making the total for the two years 8.16. Over a short period like two years the difference between simple and compound interest is of little consequence, but over a long period the difference is great.

In Life Assurance transactions Simple Interest is of no value whatever. If Interest *fall due* at certain stipulated times, and if there be the means of enforcing payment at the time when it is due, it is immaterial whether it is *called* Simple or Compound Interest. The person receiving the interest can at once re-invest it, thereby obtain interest on interest, and transform the so-called Simple Interest into Compound Interest. In like manner, if Interest be *payable* at certain fixed dates, then for all practical purposes it is Compound Interest which is being paid. Some irresponsible companies, especially building companies, have advertised that they can give great profits to their shareholders or depositors because they "pay Simple Interest and earn Compound". If they pay the interest on the due dates, they are for all practical purposes paying Compound Interest as well as earning it. No possible profit can therefore be derived from this source, and any company using such arguments must be viewed with suspicion.

The funds of Life Assurance companies are always invested at Compound Interest; that is, the interest received from investment is at once re-invested so as to earn more interest. The discussion and demonstrations hereafter given will therefore be confined to the principles of Compound Interest.

Standard Rates. The standard rate of Interest generally prescribed in America for computing Policy Values for some years prior to 1901 was 4%. But several of the principal State Insurance Departments procured the passing of new laws reducing the rate from 1st January, 1901, to 3½%. This reduced rate was applied to the new business only; while busi-

ness written on the former 4% basis continues on the same basis still. In some states higher rates are still officially sanctioned. There is no fixed rule in Great Britain as to the rate of interest to be employed in Life Assurance calculations. Companies may choose their own standards of valuation so long as they disclose the methods fully to the public. The most usual rate there adopted is 3%, while a good many British companies employ even lower rates, such as 2½%.

Effect of Standard. By using a certain rate of interest for calculating the policy values, or liability, of a Life Assurance company, the assumption is tacitly made that the funds of the company can be invested to realize that rate. If the investments realize a higher rate of interest, the additional receipts form a source of profit for the company. Keeping this situation in view, several companies in America have voluntarily adopted 3% interest as their standard, although the official rate is 3½%, on the ground that it puts them in a stronger position and enables them to earn larger margins of surplus interest in future years.

Actual Rates. The rates of interest actually earned have in the past averaged much over 4%; but during the last twenty or thirty years there has been a tendency towards reduction in the interest on first class investments; it was for this reason that the standard rates were reduced. The rate of interest on an investment is usually found by dividing a year's interest by the capital invested; but as the funds of assurance companies generally increase during the year, and the interest earnings themselves help the increase, the average rate earned cannot be obtained quite so simply. If the interest earned in any year were divided by the funds invested at the *beginning* of the year, then those companies which had a large increase in their funds would appear too favorably in the comparison. On the other hand, if the year's interest were divided by the funds at the *end* of the year the converse would hold. For measuring the interest

earned by Life Assurance companies a middle course is usually followed, and the following formula has been suggested as a good basis, namely:

$$\text{Average rate earned} = \frac{2\,I}{A + B - I}$$

In which I represents the total interest earned during the year;
A the funds at the beginning of the year; and
B the funds at the end of the year.

The reason for subtracting I from the denominator can be most clearly shown by means of a simple example: If a fund of 100 were invested at the beginning of the year at 4%, then at the end of the year it would amount to 104. If the interest were merely divided by the mean of the funds at the beginning and end of the year, we should divide 4 by 102, showing a return on the investment of *less* than 4%, which is known to be incorrect. To adjust this error half of the interest should be deducted from the mean of the funds; or, when numerator and denominator are both doubled, the entire year's interest is subtracted as in the above formula.

If this formula be used, the rates of interest earned by companies operating in the State of New York in the several years shown have been as follows:—

Year.	Average rate of Interest earned.
1880	5.4
1885	5.2
1890	5.0
1895	4.9
1900	4.5
1905	4.6

Using the same formula, the rates of interest which have been earned in Great Britain have been:—

Year.	Average rate of Interest earned.
1880	4.5
1885	4.3
1890	4.2

Year.	Average rate of Interest earned.
1895	4.1
1900	3.9
1904*	4.0

It will be observed that the reduction in the rate of interest is very noticeable in both countries. It has, however, taken place to a greater degree in the United States, and there has been a tendency for the rates in the two countries to approach each other. Any profits realized from investments have not been included; and taxes have not been deducted from the interest in either case, although in Great Britain this charge is definitely incurred against the interest income. In the United States the taxes, which are heavy, are in some states levied against the reserve values of the policies and in others against the premiums collected.

NATURE OF INVESTMENTS.

Legal Requirements. In the United States and Canada assurance companies are permitted to hold only certain classes of investments. In some states there are no special provisions as to investments; but in such cases the Insurance Commissioner has generally a discretionary power and must "be satisfied". Naturally he adopts rules similar to those employed elsewhere. A brief summary may therefore be given of the investments authorized by the laws of New York, as these indicate generally the view taken by legislatures as to the securities which are suitable for a Life Assurance company. Prior to 1906 these were as follows:—

(a) Stocks or bonds of the United States or of the State of New York.

(b) Bonds of county or incorporated city in the State of New York, not above par or market value.

(c) First bonds or mortgages on improved real estate worth 50 per cent more than the amount loaned, subject to buildings being insured against fire.

* This is the latest rate available. It is computed from the Board of Trade Returns published in 1906, giving the figures rendered during the year 1905. Most of these figures relate to the calendar year 1904.

(d) Stocks and bonds of solvent institutions incorporated under the laws of the United States or of any state.
(e) Real estate acquired for certain defined purposes or under foreclosure.
(f) Bonds of any city, incorporated town, village, or school district of the State of New York, provided two-thirds of the directors approve.
(g) Loans on Life policies, within the reserve.
(h) Corporations doing business in other states may invest in such states, in conformity with the laws thereof.

Additional restrictions have been imposed by the new laws of 1906. Stocks and shares of all kinds are prohibited, and present holdings of such securities must be sold within five years. Real Estate cannot be acquired as an investment, and if not required for office accommodation, must be sold within five years from date of acquisition. Collateral Trust Bonds, issued on the security of stocks, are excluded, and all Syndicate and Joint-Account transactions (even for the underwriting of first mortgage bonds) are prohibited.

Securities must be interest-bearing. Loans on personal security, bills, and other obligations of like nature are not admitted as assets; in like manner, "Agents' Balances" and "Office Furniture" are excluded.

Investments of British Companies.

In Great Britain the situation as regards the investment of funds is entirely different. Companies are unrestricted except by their own constitution or charter, but the nature of the investments must be fully disclosed; the general principle there followed is entire freedom of action so long as there is complete publicity regarding such action. The result is that Life Assurance companies in Great Britain have invested to a considerable extent in foreign securities, particularly the bonds of foreign governments and the securities of first class railroads, and some have even placed considerable portions of their funds in mortgages on real estate in foreign countries. Personal security loans, agents' balances, office furniture, and other assets of like nature, including even,

in the case of some young companies, expenses of organization, appear in their lists of assets. It is there usual to have an independent audit of the annual accounts of each company, and a certificate granted by the auditor, who is generally a Chartered Accountant, or what is known in the United States as a Public Accountant. He certifies that the published accounts are correctly stated, and that all his requirements in examining the statements have been complied with. There is now a tendency to follow this practice in America. Several companies have employed independent experts, and it has even been suggested that such an audit should be made compulsory by law.

INTEREST TABLES.

While it may be very desirable to know the theory for calculation of compound interest, and while the figures may be computed individually in accordance with the formulas hereafter explained, nevertheless for practical purposes printed tables of Compound Interest give all the information generally required by a simple reference and without the trouble of calculation. The tables commonly available are:—

1. *The amount of 1* accumulated for a long period of years at various rates of interest, the *formula* being $(1+i)^n$
2. *The present value of 1* due any number of years hence at various rates of interest; *formula* $v^n = \dfrac{1}{(1+i)^n}$
3. *The amount of 1 per annum* accumulated for any number of years at various rates of interest; *formula* $S_{\overline{n|}} = \dfrac{(1+i)^n - 1}{i}$
4. *The present value of 1 per annum* payable at the end of each year; *formula* $a_{\overline{n|}} = \dfrac{1 - v^n}{i} = \dfrac{1-(1+i)^{-n}}{i}$
5. *The annual payment for any number of years which 1 will purchase; formula* $\dfrac{1}{a_{\overline{n|}}} = \dfrac{i}{1 - v^n}$

From such tables any kind of interest calculation can conveniently and rapidly be made, and excellent tables have been

computed on several occasions, in particular those published in Vienna by Professor Simon Spitzer. They give each of the above functions from 1 to 100 years at rates of interest from 1/8% to 15%, at intervals of 1/8 and 1/6 up to 6%, thereafter at intervals of 1/4 and 1/3 to 10%, and from 10% to 15% at intervals of 1/2%. The figures are given to eight decimal places, so that very accurate calculations can be made.

When interest on 100 is payable half-yearly at a specified rate per cent, one half of that rate is due at the end of six months, and when this has been received it may of course be reinvested for the remaining six months so as to earn more interest. When interest is payable quarterly the sums received are available for reinvestment at the end of each three months. It is apparent therefore that the receipt of interest half-yearly or quarterly secures a slightly better return than if it were due yearly. The reason for subdividing the published interest rates into such small intervals as those above mentioned is not that those very low rates are either paid or contracted for in practice, but simply that calculations on half-yearly and quarterly bases may be readily made.

Practical Illustration. If we wish to find the amount of 100 in ten years at 3½%, the result is obtained at once from the first interest table above mentioned (see Table I. in the Appendix) under the heading 3½% and opposite the tenth year. The result is **141.06**. If it be desired to find the amount at 3½% *payable half-yearly*, the result can be obtained by referring to the table under 1¾% opposite the twentieth year, the result being **141.48**. There are really twenty compounding periods in ten years when the interest is payable half-yearly, and each semi-annual payment is at the rate of 1¾%; hence it is necessary in using the table for half-yearly interest to take the same amount at one-half the rate of interest per annum, and for double the number of periods. In like manner the amount of 100 at 3½% per annum *payable quarterly* would be obtained under the table for 7/8% opposite the year forty, there being forty compounding periods in such case. The result is **141.69**.

Similar methods apply to the present value of any sum due in future; also to the amount and present value of annual payments. When an annual payment is being dealt with, it is necessary not only to double or quadruple the number of years, but at the same time to take one-half or one-quarter of the annuity payment before multiplying. An example will most readily illustrate this:—To find the present value of an annuity of 20 for ten years payable in quarterly instalments of 5 each, at 3½% interest, we must refer to the table of the present value of 1 per annum at rate of interest ⅞% under the period forty years, and thereafter multiply by 5. The result is **168.14,** as will be seen on reference to Table IV. in the Appendix. Similar methods may be applied in the case of other tables.

The following is another example common in practice and combining the use of two tables:—

Find the purchase price of a 4% bond of 1,000 running for ten years with half-yearly interest, so that the purchaser may earn 3½% on his investment.

The purchaser receives two distinct benefits:—

(1) 1,000 at the end of ten years, and
(2) 20 each half year for ten years,
i. e.:—for 20 payments altogether.

(1) The present value of 1,000 due in ten years at 3½%, payable half-yearly, is found by referring to a table of v^n (Table II.) at 1¾% for a period of twenty units of time, that is: $1000\, v^{20} = 706.82$

(2) The value of 20 each half year for ten years (involving 20 payments altogether) will be found under a table of $a_{\overline{n|}}$ (Table IV.) at 1¾%, that is:.......................... $20\, a_{\overline{20|}} = 335.06$

The Total Price, therefore, which a purchaser could afford to pay so as to realize 3½% would be.... 1,041.88

CHAPTER V.

INTEREST AND ANNUITIES-CERTAIN.—TECHNICAL.

Rate per Unit. In speaking of the *Rate* of Interest it is usual to refer to the "rate per cent per annum"; but on investigating the theory of the subject it is more convenient to use the "*Rate per unit per annum*", which may for convenience be represented by the letter i. It should be very particularly noticed that when the rate is 5%, then i, the rate per unit, is .05; when the rate is 2½%, i is .025, etc. It might be still further generalized by treating of the rate of interest on each unit invested for each unit of time, but a year is a convenient time unit. If 100 were invested at 5% for one year, the total sum including interest would be 105 at the end of the year. Similarly, if 1 were invested at rate i, the amount would be $(1+i)$; and, if i were taken at 5%, the amount would be 1.05. In business affairs the rate per cent is always used, but for mathematical analysis the rate *per unit* is simpler.

Accumulation at Interest. If 1 be invested for a year at rate i the amount at the end of the year is $(1+i)$, and if any other sum, P, were invested at the same rate the amount at the end of the year would be $P+Pi$, or $P\ (1+i)$. The letter P is used as being the initial letter of Present Value; but the same letter is afterwards used to denote Premium, and the meanings must therefore be carefully distinguished: usually, however, the letter has a subscript when used to indicate a premium—as P_x or ${}_nP_x$. If at the end of the first year the sum $(1+i)$ were invested at the same rate, the amount at the end of the second year would be $(1+i)\ (1+i)$, or $(1+i)^2$; similarly, if $P\ (1+i)$ were invested for a year it would amount to $P\ (1+i)^2$. But to assume an actual reinvestment each year is unnecessary

when we speak of Compound Interest, where reinvestment is understood as a matter of course. Therefore we may say that if 1 be invested for three years at rate of interest i the amount will be $(1+i)\ (1+i)\ (1+i) = (1+i)^3$, and in like manner if 1 be invested for n years the amount will be $(1+i)^n$. If the sum invested be P, the amount in n years will be $P\ (1+i)^n$. If we express this accumulated amount by the letter S (the initial letter of Sum) we may write this result in the form of an equation:—

$$S = P\ (1+i)^n \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

This equation is of great importance. It may be mathematically transformed as follows:—

$$P = \frac{S}{(1+i)^n} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\text{Or again } (1+i)^n = \frac{S}{P}$$

$$\text{whence } (1+i) = \sqrt[n]{\frac{S}{P}} \text{ or } \left(\frac{S}{P}\right)^{\frac{1}{n}}$$

$$\text{and } i = \left(\frac{S}{P}\right)^{\frac{1}{n}} - 1 \qquad \ldots\ldots\ldots\ldots(3)$$

$$\text{Or, as } (1+i)^n = \frac{S}{P}$$

$$n \log (1+i) = \log S - \log P$$

$$\text{and } n = \frac{\log S - \log P}{\log (1+i)} \qquad \ldots\ldots\ldots\ldots(4)$$

Students should observe particularly that this equation (4) is an actual division of logarithms. If the process of dividing were to be performed *by* logarithms the equation would be expressed:

$$\log n = \log\{\log S - \log P\} - \log\ \{\log (1+i)\}$$

whence find n by entering the logarithmic table inversely; or from a table of anti-logs.

If, therefore, any three of the four quantities, S, P, i and n, be known the fourth may always be found. *For example*:—If we know that for 300 paid now the sum of 382.88 will be received in five years, then to find the rate of interest which such a transaction would realize we have —

Present value.........$P = 300$
Amount..............$S = 382.88$
Number of years......$n = 5$

The rate of interest... $i = \left(\frac{S}{P}\right)^{\frac{1}{n}} - 1$

$$= \left(\frac{382.88}{300}\right)^{\frac{1}{5}} - 1 = \sqrt[5]{1.2763} - 1$$
$$= \quad 1.05 - 1$$
$$= \quad .05$$

Or, to use the general phrase, the rate of interest is 5%.

In practice such questions are more frequently solved by the use of interest tables.

Money Doubles Itself. One useful and practical application of the formula for finding n may with advantage be discussed, namely: *to find the time in which money will double itself at Compound Interest.* In this case if $P=1$ then of necessity $S=2$ so as to fulfil the conditions, and formula (4) becomes

$$n = \frac{\log 2 - \log 1}{\log (1+i)}$$

By using natural logarithms to the base e we can find a very convenient rule, as follows:—

$$\log_e 2 = .693$$
$$\log_e 1 = 0$$
$$\log_e (1+i) = i\left(1 - \frac{i}{2} + \frac{i^2}{3} \ \&c.\right)$$
$$\therefore \quad n = \frac{.693}{i\left(1 - \frac{i}{2} \ \&c.\right)}$$

Now i has a very small value, and we may therefore neglect the higher powers of i in the above formula, and write

$$n = \frac{.69}{i} \text{ approximately.}$$

From this result we get the common rough rule: *Divide 69 by the rate per cent.*

By extending the above formula to a second term a more accurate rule is obtained:—

$$n = \frac{.693}{i\left(1 - \frac{i}{2}\right)} = \frac{.693}{i}\left(1 + \frac{i}{2}\right)$$

$$= \frac{.693}{i} + .347$$

The more accurate rule is therefore:—

To find the time in which money doubles itself at Compound Interest, Divide 69 by the rate per cent, and add .35.

Results by this rule would be as follows, the exact times being added to show how close the approximations are:—

TIME IN WHICH MONEY DOUBLES ITSELF AT COMPOUND INTEREST.

Rate per cent.	Approximate Time by rule.	Exact Time.
3	23.35 years	23.45
4	17.60 "	17.67
5	14.15 "	14.21

If 69.3 were used in the rule instead of the integral number 69, the approximate results would be almost exact to the second decimal place.

There are two other symbols which are much used in interest calculations, namely:—

v = the present value of 1 due one year hence; and

d = the discount on 1 due in one year.

The amount of 1 in one year being $(1+i)$, it follows that the sum which, invested at the beginning of the year, would produce 1 at the end would be $\frac{1}{1+i}$, which is the present value of 1 due one year hence, and is usually represented by the letter v. The present value of 1 due n years hence would be $\frac{1}{(1+i)^n}$, and therefore we have the equations:—

$$v = \frac{1}{(1+i)}, \text{ or } (1+i)^{-1} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

$$\text{and } v^n = \frac{1}{(1+i)^n}, \text{ or } (1+i)^{-n} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

DISCOUNT.

Discount is the name applied to the charge made in advance for the use of money to be repaid at some future time. For example, if a man were to sign a note agreeing to pay 100 one

year hence, and a banker were to buy this note at 5% discount, the latter would pay 95 for the note, and receive 100 at the end of the year. Now it is apparent that this transaction is different from that of charging interest at 5%. The *interest* at 5% on the 95 paid by the banker would be 4.75, so that if he were to receive only 99.75 at the end of the year he would have earned 5% *interest* on his money. As the bargain is that he should receive 100, it is evident that he actually earns more than 5% interest.

Bankers' and True Discount. When the charge for the use of money is made in the above form it is distinguished by some writers as "Commercial Discount", or "Bankers' Discount"; while they at the same time speak of the interest on the sum actually paid as "True Discount". To illustrate: the present value of 100 in one year at 5% interest is $\frac{100}{1.05}$ or 95.24. If, therefore, the banker were to purchase the note for 95.24 he would receive 5% interest on his outlay, and therefore (100 — 95.24), or 4.76, would be called the "True Discount" on 100. It is also the interest at 5% on 95.24.

But it seems unnecessary to make this distinction between "Bankers' Discount" and "True Discount". The real distinction lies between Discount and Interest; and, when a banker speaks of charging *discount* at 5%, it should be, and is, understood that this differs from charging interest at 5%. True Discount has, however, been discussed in most text-books on the subject, and is represented by the interest on the present value of money. The True Discount for a single year would be the difference between the sum of 1 due at the end of the year and its present value. This is represented by d, so that we have

$$d = 1 - v \qquad \text{(7)}$$

$$= 1 - \frac{1}{(1+i)}$$

$$= \frac{1 + i - 1}{1 + i}$$

$$= \frac{i}{1+i} \qquad \text{(8)}$$

$$= i\left(\frac{1}{1+i}\right) = iv \qquad \text{(9)}$$

From formula (7) also we may write

$$v = 1 - d \qquad \text{....... ....(10)}$$

The formulas (7) and (9) give two distinct views of this question of discount, the former being the "difference between the sum due and its present value", and the latter the "interest on the present value". In the numerical example above discussed these two formulas are illustrated, because

$$100 - 95.24 = 4.76 = .05 \times 95.24$$

HALF-YEARLY AND QUARTERLY INTEREST.

The calculation of interest by using yearly intervals is a convenient method; but any other interval, or unit of time, would serve the purpose equally well. In practice interests are most frequently payable yearly, half-yearly, or quarterly; but though thus payable, the rate is still quoted as the "rate per cent per annum". If the rate of interest were 5% per annum, payable half-yearly, the actual sums which would fall due on a loan of 100 would be 2.5 at the end of the first half-year, 2.5 at the end of the second half-year, and the same amount at the end of each half-year thereafter. Accordingly, the simplest method of investigating this question of half-yearly interest would be to deal with interest at the rate of 2.5 and change the unit of time from one year to one half-year, as was explained in the last chapter. For half-yearly interest therefore, if the symbols retain the same meanings as before, the formulas would become

$$S = P\left(1 + \frac{i}{2}\right)^{2n}$$

$$P = \frac{S}{\left(1 + \frac{i}{2}\right)^{2n}}$$

$$n = \frac{\log S - \log P}{2\log\left(1 + \frac{i}{2}\right)}, \text{ and}$$

$$i = 2\left\{\left(\frac{S}{P}\right)^{\frac{1}{2n}} - 1\right\}$$

In like manner, if the interest were computed at 5% and payable quarterly the actual payments would be 1.25 at the end of the

first quarter, 1.25 at the end of the second quarter, and so forth throughout the duration of the loan. This becomes equivalent to calculating interest at the rate of 1.25, and treating the unit of time as a quarter of a year. The general formula would be

$$S = P\left(1 + \frac{i}{4}\right)^{4n}$$

and the others would be similarly modified.

The question may be made entirely general by assuming that interest is payable m times a year, when of course any value may be given to m; and the formulas become

$$S = P\left(1 + \frac{i}{m}\right)^{mn} \qquad \text{..................(11)}$$

$$P = \frac{S}{\left(1 + \frac{i}{m}\right)^{mn}} \qquad \text{.................(12)}$$

$$n = \frac{\log S - \log P}{m \log\left(1 + \frac{i}{m}\right)} \qquad \text{..............(13)}$$

$$\text{and } i = m\left\{\left(\frac{S}{P}\right)^{\frac{1}{mn}} - 1\right\} \qquad \text{...........(14)}$$

The extreme limit for the value of m might be taken, thus assuming that interest is payable continuously at infinitesimal intervals, but the resulting formulas pass beyond the scope of this work.

NOMINAL AND EFFECTIVE RATES OF INTEREST.

If interest be at the rate of 5% per annum, and be payable half-yearly, it is at once evident that on receiving payment of the interest at the end of the first half-year the amount may be re-invested for the second half-year, and will itself earn more interest. Dealing with the same example as before, if 2.5 be received for interest at the end of the first half-year, and if it be re-invested at the same rate of 5% for the second half-year, there will be received .0625 additional interest, so that the entire proceeds from the 100 lent during the first year would not be 5, but 5.0625, made up thus: 2.5 + 2.5 + .0625. Although, therefore, the basis of calculation may be 5%, yet from the fact that in-

terest is payable half-yearly the actual sum realized is somewhat greater. This interest actually realized is generally called the *effective rate;* while the rate used for purposes of calculation is called the *nominal rate.*

If the symbol j be used for the nominal rate of interest, while the symbol i be retained for the effective rate, the relationship between the two may be expressed in general terms. If interest be payable m times a year, the amount of 1 at rate j in one m^{th} part of a year would be $\left(1+\frac{j}{m}\right)$; in two m^{th} parts $\left(1+\frac{j}{m}\right)^2$; and in a full year (that is, in m units) it would be

$$\left(1+\frac{j}{m}\right)^m$$

Deducting the principal of 1, we get the actual interest realized in a year, namely:—

$$i^{(m)} = \left(1+\frac{j}{m}\right)^m - 1 \qquad \ldots\ldots\ldots\ldots(15)$$

The formula may of course be mathematically transposed so as to express the *nominal* rate in terms of the *effective;* but this form of the question is seldom of practical value although a good exercise in algebra. When interest is payable half-yearly the above formula becomes

$$i^{(2)} = \left(1+\frac{j}{2}\right)^2 - 1$$

and when quarterly

$$i^{(4)} = \left(1+\frac{j}{4}\right)^4 - 1$$

The effective rates of interest corresponding to the nominal rates most frequently in use are as follows:—

NOMINAL RATES %	EFFECTIVE RATES %		
	Yearly.	Half-Yearly	Quarterly.
2.5		2.516	2.524
3		3.023	3.034
3.5	Same	3.531	3.546
4	as	4.040	4.060
5	Nominal	5.063	5.095
6		6.090	6.136

Annuities-Certain.

An Annuity in its literal sense is simply a yearly payment; but it has gradually come to be understood that any payments made at periodic intervals form an annuity, and some such contracts provide for monthly and even weekly payments. Generally the annuity is measured by the total of the payments throughout the year, that period being used as the unit of measurement. If the annuity payments depend upon the survival of any person the contract is called a Life Annuity; and if the payments are to be made for a fixed period, irrespective of any life, it is called an Annuity-Certain.

An Annuity-Certain implies an obligation to pay a definite amount of money each year for a fixed period of time. Dealing first with the simplest general case under which a payment of 1 would be made at the end of each year for n years, we can find, as was done when considering interest on a single payment, the present value and the accumulated amount of such payments. The present value is represented by the symbol $a_{\overline{n}|}$, where the subscript n denotes the number of years, and the bracket $\overline{}|$ indicates that the period is definite or certain. As the series of payments of 1 each are due at the end of the 1st, 2nd.... years, we have

$$a_{\overline{n}|} = v + v^2 + v^3 + \dots\dots\dots + v^n$$

Summing this geometrical series we have

$$a_{\overline{n}|} = v\,\frac{1-v^n}{1-v} = \frac{1-v^n}{(1+i)-(1+i)\frac{1}{1+i}}$$

$$= \frac{1-v^n}{i} \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(16)$$

Annuity Due. The above formula considers the first payment of the annuity to be at the *end* of the year; but occasionally the first payment is made at the *beginning*. If the annuity be 1 for n years with the first payment due at once, then it is evident

that the effect is that the present value will consist of the full value of the first payment, followed by an ordinary annuity running for one year less, that is for (n—1) years. When the annuity begins at once it is called an *annuity due*, and the symbol is written $\mathbf{a}_{\overline{n}|}$. We therefore have

$$\mathbf{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|} \quad \text{..................(17)}$$

Deferred Annuity. The first payment of an annuity might also be *deferred* for a number of years. If it were deferred for m years the symbol would be written $_{m|}a_{\overline{n}|}$, and this means that at the end of m years the value would be $a_{\overline{n}|}$. In other words, the first payment of the annuity would be delayed for yet another year, so that when the annuity is spoken of as being deferred for m years it is understood that the first payment will take place at the end of (m+1) years. As the value of the annuity at the end of m years is $a_{\overline{n}|}$, the value at the present time must be

$$_{m|}a_{\overline{n}|} = v^m \, a_{\overline{n}|} \quad \text{..................(18)}$$

This result might have been obtained by considering each payment of the deferred annuity in the same way as was done with the immediate annuity, namely:—

$$_{m|}a_{\overline{n}|} = v^{m+1} + v^{m+2} + \ldots\ldots\ldots\ldots + v^{m+n}$$

$$= v^m (v + v^2 + \ldots\ldots\ldots\ldots + v^n)$$

$$= v^m \, a_{\overline{n}|} \quad \text{as before.}$$

Or we may deal with it in yet another way, which furnishes an interesting formula:—

$$_{m|}a_{\overline{n}|} = v^{m+1} + v^{m+2} + \ldots\ldots\ldots\ldots + v^{m+n}$$

If we add to and subtract from this the series

$$v + v^2 + \ldots\ldots + v^m$$

the result will not be affected and we have

$$
\begin{aligned}
{}_{m|}a_{\overline{n}|} &= v + v^2 + \ldots\ldots + v^m + v^{m+1} + \ldots\ldots + v^{m+n} \\
&\qquad - (v + v^2 + \ldots\ldots + v^m) \\
&= a_{\overline{m+n}|} - a_{\overline{m}|} \qquad \ldots\ldots (19)
\end{aligned}
$$

a result which at once commends itself to the reader as being logical and correct.

If it were desired that the first payment of a deferred annuity be made at the end of m years, instead of one year later, the formulas would be

$$
\begin{aligned}
{}_{m|}\mathbf{a}_{\overline{n}|} &= v^m\, \mathbf{a}_{\overline{n}|} \\
&= \mathbf{a}_{\overline{m+n}|} - \mathbf{a}_{\overline{m}|}
\end{aligned}
$$

Perpetuity. In formula (16) for the present value of an annuity v^n becomes smaller and smaller with increasing values of n; and, in the limit, when n is made infinitely great v^n becomes equal to zero. We therefore have as the value of an annuity to run for ever, known as a *perpetuity*,

$$a_\infty = \frac{1-0}{i}, \text{ or } \frac{1}{i} \qquad \ldots\ldots (20)$$

This result could have been obtained by general reasoning; because, if 1 be permanently invested at rate i, it will produce i each year for ever, and therefore 1 is the present value of a perpetuity of i. By simple proportion, therefore, the present value of a perpetuity of 1 must be $\frac{1}{i}$, as before.

The value of a perpetuity-due is

$$
\begin{aligned}
\mathbf{a}_\infty &= 1 + a_\infty \\
&= 1 + \frac{1}{i} = \frac{1+i}{i} \\
&= \frac{1}{d} \qquad \ldots\ldots (21)
\end{aligned}
$$

Amount of Annuity Certain. By a process similar to that used for the Present Value, the accumulated amount of an Annuity-Certain for n years may be found, and this is represented by the symbol $S_{\overline{n}|}$. As the first annuity payment is

made at the end of the first year, and accumulates at interest until the end of the n^{th} year, the amount of this one payment will be $(1+i)^{n-1}$; the second payment will amount to $(1+i)^{n-2}$; and so on until we come to the last payment, which is made just at the date the accumulation ends, and is therefore 1. The whole series of payments, therefore, accumulated with interest, will be

$$S_{\overline{n|}} = (1+i)^{n-1} + (1+i)^{n-2} + \ldots\ldots + (1+i) + 1$$

Summing this geometrical series we have

$$S_{\overline{n|}} = \frac{(1+i)^n - 1}{(1+i) - 1} = \frac{(1+i)^n - 1}{i} \quad \ldots\ldots\ldots\ldots (22)$$

It will be apparent that the result should be the same whether we accumulate each individual payment, or take the present value at the commencement of the annuity and accumulate the present value for the period of n years. We can prove that this is so mathematically:—

$$a_{\overline{n|}} = \frac{1 - v^n}{i}$$

If this be accumulated for n years the result will be

$$\frac{1 - v^n}{i} \times (1+i)^n = \frac{(1+i)^n - 1}{i}, \text{ which is } S_{\overline{n|}}$$

In like manner the present value of the accumulation should be the same as the sum of the present values of all the payments. The value now of $S_{\overline{n|}}$ due in n years is

$$v^n \times \frac{(1+i)^n - 1}{i} = \frac{1}{(1+i)^n} \times \frac{(1+i)^n - 1}{i}$$

$$= \frac{1 - \dfrac{1}{(1+i)^n}}{i}$$

$$= \frac{1 - v^n}{i} \quad \text{or } a_{\overline{n|}} \text{ as before.}$$

CONSTRUCTION OF TABLES.

Continuous Process. In constructing tables of interest, or for that matter in constructing general forms of tabular matter, it is desirable to adopt what is known as a continuous

process; that is, to derive one value from the value immediately before it. The great advantage of such a process is that if any particular value be checked independently and be found correct, it is at once known that all values prior to this specific value are in like manner correct.

One of the simplest examples of calculation by continuous process is to form a table expressing the number of dollars equivalent to pounds sterling. For example if £1 be equivalent to \$4.8665, and it be required to find the value in dollars of £2, £3, etc., the values can be computed by multiplication, but an error might be made in any one of these and each would have to be separately checked to make sure that the table was correct. On the other hand, if the table were constructed *by addition* and the tenth value were found to be correct, then *all* must be correct. The process *by addition* is to write down the first value 4.8665, write the same amount on a loose slip, add it to the first value and obtain the second value 9.7330. Moving down the slip of paper so as to cover the first figure, 4.8665 should now be added to 9.7330, giving the third value 14.5995. In like manner, subsequent values are formed, until the tenth value should appear as 48.6650. If this is the tenth value obtained by addition then all ten are correct, and no checking is necessary.

Table of Amounts. In like manner, in forming interest tables the simplest case may be discussed, namely: $(1+i)^n$. Values can be independently computed for each year, but it is much better to begin with the first value $(1+i)$, and form the entire table by multiplying the preceding value by $(1+i)$. Then each twentieth value can be checked by direct calculation by means of logarithms or otherwise. A numerical example of such calculation at 3½% may be of value, contracted multiplication being used:—

	1.035	......$(1+i)^1$
Multiplying by .03	3105	
" " .005......	5175	
	1.071225	......$(1+i)^2$

[Carried up]	1.071225	......$(1+i)^2$
	321368	
	53561	
	1.1087179	......$(1+i)^3$
	332615	
	55436	
	1.1475230	......$(1+i)^4$
	344257	
	57376	
	1.1876863	......$(1+i)^5$
	356306	
	59384	
	1.2292553	......$(1+i)^6$
	etc.	etc.

In forming a table of v^n the final process is exactly the same; but, as it is easier to multiply than to divide, it is better first to calculate accurately the *last* value in the table, and thereafter *multiply* constantly by $(1+i)$. This process is better than beginning with the *first* value and *dividing* by $(1+i)$.

From the foregoing it will be seen that in calculating tables by means of a continuous process it is necessary to have:—

1. The initial value;
2. A working formula; and
3. A formula for independent verification.

Tables of $S_{\overline{n}|}$ and $a_{\overline{n}|}$ may in like manner be computed by the continuous process by the following formulas:—

$$S_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

$$\therefore\ (1+i)\,S_{\overline{n}|} = \frac{(1+i)^{n+1} - (1+i)}{i} = \frac{(1+i)^{n+1} - 1}{i} - 1$$

$$= S_{\overline{n+1}|} - 1$$

whence $S_{\overline{n+1}|} = (1+i)\,S_{\overline{n}|} + 1$ ....(23)

$$\text{And } a_{\overline{n}|} = \frac{1 - v^n}{i}$$

$$\therefore (1 + i)\, a_{\overline{n}|} = \frac{(1 + i) - v^{n-1}}{i}$$

$$= \frac{1 - v^{n-1}}{i} + 1 = a_{\overline{n-1}|} + 1$$

$$\text{whence } a_{\overline{n-1}|} = (1 + i)\, a_{\overline{n}|} - 1 \qquad \text{..........(24)}$$

The process in the latter case therefore contemplates the calculation of the last value in the table, the continuous multiplication by $(1+i)$, and the subtraction of 1 each time.

The above are the simplest and most easily understood arithmetical methods for the calculation of tables; and they are probably sufficient to explain the nature of the tools provided for the professional workman.

CHAPTER VI.

PROBABILITY, PROBABILITIES OF LIFE, AND EXPECTATION OF LIFE.

Definition and Illustration. Probability is the science of measuring chances. Its nature will be best understood by means of a very simple example:—If in a bag there are two coins of exactly the same shape and size, one copper and the other silver, and a person is allowed to put his hand into the bag and take one of the coins, it is an even chance as to whether he takes the copper or the silver coin. If there were two copper and two silver coins, it would still be an even chance as to whether a silver coin would be drawn. If on the first drawing a copper coin were taken out and a second drawing were permitted, then, as there would be two silver coins and one copper coin in the bag, the chance of drawing a silver coin as compared with the chance of drawing a copper coin on the second trial would be as 2 : 1.

Probability a Fraction. The most convenient way of representing such conditions mathematically is by means of fractions, and in the last example the probability of drawing a silver coin would be expressed as $\frac{2}{3}$, and the probability of drawing a copper coin as $\frac{1}{3}$. When the contents of the bag are known and the opportunity is given of drawing a coin, there is a certainty of drawing either a copper or a silver coin, represented by the sum of the two probabilities: $\frac{2}{3} + \frac{1}{3} = 1$. When, therefore, an event is certain to occur, the probability is represented by 1. This is a convenient result, and one which makes the study of the subject more easy. When the chances are even, the probability is written as $\frac{1}{2}$.

Algebraic Definition. Taking up algebraically the general case:—If an event can happen in a ways and fail in b ways, and each of these ways is equally likely, the *probability*, or the *chance*, of its happening is $\frac{a}{a+b}$; and in like manner the probability of its failing is $\frac{b}{a+b}$.

As the event must either happen or fail, we may represent the happening of the event by $1 - \frac{b}{a+b}$ and the failing of the event by $1 - \frac{a}{a+b}$. These forms are mathematical equivalents of the others, because

$$1 - \frac{b}{a+b} = \frac{a+b-b}{a+b} = \frac{a}{a+b}; \text{ and}$$
$$1 - \frac{a}{a+b} = \frac{a+b-a}{a+b} = \frac{b}{a+b}$$

Events are said to be "equally likely" when we have no reason to expect any one rather than any other. For instance, in the above example we have just as much reason to expect a copper coin to be drawn as a silver one. But there is another meaning which may be given to "equally likely"—events may be said to be "equally likely" when they recur with regularity *in the long run*. And the probability of an event occurring in such case is the ratio of the number of times in which it occurs in the long run to the sum of the number of times in which events of that description occur and the number of times in which they fail to occur. Life Assurance Probabilities are estimated by finding the ratio of the actual number of times the event occurs in a large number of cases to the whole number of times in which it occurs and in which it fails.

Two Independent Events. *The probability that of two events, independent of each other, both will occur is the product of their separate probabilities.*

In tossing a coin the probability of its turning up a head is an even chance, or ½; what is the probability of turning up a head each time in two trials? By the above rule the answer will be

$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. This can be found as follows: the number ot ways in which the coin may lie are four, namely:—

(1) Head first time; head second time
(2) Head " " tail " "
(3) Tail " " head " "
(4) Tail " " tail " "

But only the first of these four ways will satisfy the question, and therefore the probability is $\frac{1}{4}$, or as 3:1 against.

Algebraically the question may be treated thus:—If the first of the two independent events may happen or fail in r ways altogether, and the second in s ways, then if these ways are all equally likely, and all independent, the total number of possible ways must be $r \times s$, because each one of the r ways may be associated with each one of the s ways. If a of these ways would fulfil the required conditions in the first event, and b of them in the second event, then the total number of ways in which both conditions *could* be fulfilled would be $a \times b$; seeing that, as before, each one of the a favorable cases could be associated with each one of the b cases. The probability of the occurrence of both events would therefore be $\frac{ab}{rs}$ or, what is the same thing, $\frac{a}{r} \times \frac{b}{s}$

PROBABILITIES OF LIFE.

The foregoing simple definitions and general explanations of Probability are sufficient as a foundation for the science of Life Assurance. Further elaboration would be useful to students who are interested in the subject, but these propositions seem to be the *essentials* for an understanding of the simpler questions which arise.

Probability and Mortality. It is necessary in life assurance transactions to take into consideration the probability of living or dying in any year and the present value of money at interest. The probability of living or of dying is usually found by means of a mortality table, but the mortality table and such probabilities are so closely connected with one another that if the one be known the other can be calculated.

As a basis it is sufficient to have *either* the mortality table in the form afterwards explained, *or* the probabilities of living.

Mortality Tables Used. A mortality table has been defined as an "instrument by means of which are measured the probabilities of life, and the probabilities of death". If we could trace 100,000 people from the date of their birth until the date of death in each case, and record the number dying in each year of age, and the number living at the end of each year, we should have a "Mortality Table". It is however impracticable to trace any fixed group of persons in such a way, and several practical methods have been devised for arriving at equally useful results. The important consideration is the *relative* number at each age as compared with the number alive at the preceding age—the actual numbers are of little consequence.

The number living at any age x is represented by l_x; thus the number living at age 10 would be l_{10}, and at age 30, l_{30}; while the number dying in the year of age from x to $x+1$ is represented by d_x, the l and the d being of course the initial letters of living and dying. Thus d_{30} represents the number of persons who die between ages 30 and 31 out of l_{30} persons who have attained age 30.

The following schedule gives the form of such a table, as usually published. The figures are those for the youngest and oldest ages dealt with by the

AMERICAN EXPERIENCE TABLE OF MORTALITY.

Age.	Number living.	Number dying.	Probability of living a year.	Probability of dying within a year.
x	l_x	d_x	p_x	q_x
10	100,000	749	.99251	.00749
11	99,251	746	.99248	.00752
12	98,505	743	.99246	.00754
13	97,762	740	.99243	.00757
14	97,022	737	.99240	.00760
..	......	...	......	......
90	847	385	.54545	.45455
91	462	246	.46753	.53247
92	216	137	.36574	.63426
93	79	58	.26582	.73418
94	21	18	.14286	.85714
95	3	3	.00000	1.00000

As the numbers living decrease with the increase in age, it follows of necessity that l_{x+1} is less than l_x. The decrease is caused solely by deaths, and therefore

$$d_x = l_x - l_{x+1} \qquad \ldots\ldots\ldots\ldots\ (25)$$

$$\text{whence } l_x = l_{x+1} + d_x$$

In the same way the deaths over any number of ages may be found by deducting the number living at the end of the time from the number living at the beginning. The deaths over two years x and $x+1$ would be $d_x + d_{x+1}$ and we have

$$l_x - l_{x+2} = d_x + d_{x+1}$$

and similarly for a period of n years

$$l_x - l_{x+n} = d_x + d_{x+1} + d_{x+2} + \ldots\ldots + d_{x+n-1}$$

If such a large value be given to n that l_{x+n} falls outside the limits of the table, and is therefore equal to zero, we obtain the formula

$$l_x = d_x + d_{x+1} + d_{x+2} + \ldots\ldots \text{ to end of table} \qquad (26)$$

This formula does not require proof because we know that every person living must die, so that the sum of the deaths from any age to the end of the table must equal the number living at that age.

Numerical Illustration. From the above table it will be observed that the number living at age 10 is given as 100,000. This is called the "radix" of the table, and it is an assumed number. It does not mean that 100,000 actual specified persons were observed and traced from age 10 upwards, although the final placing of the figures in this form attains practically the same result. Out of the 100,000, 749 died between ages 10 and 11; therefore the *probability of aying* in the year is 749 ÷ 100,000, or .00749. The number living at age 11 is 99,251, and therefore the *probability of living* a year is 99,251 ÷ 100,000, or .99251. The probability of living a year is expressed as p_x, so that in the case discussed p_{10} equals .99251. p is the initial letter of the word probability, and the probability of dying is expressed by q_x, q being the next letter to p. In accordance with the general principles of probability, it follows that, as one must either live or die,

$$p_x + q_x = 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(27)$$

$$p_x = 1 - q_x \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(28)$$

$$q_x = 1 - p_x \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(29)$$

The above numerical example will be observed to follow these rules.

To express the matter generally we have

$$p_x = \frac{l_{x+1}}{l_x} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(30)$$

transforming this equation

$$p_x \times l_x = l_{x+1}$$

and $$\frac{l_{x+1}}{p_x} = l_x$$

The probability of living for two years is expressed as ${}_2p_x$ and the probability of living for n years as ${}_np_x$. From the above table we can readily deduce ${}_2p_{10}$, because out of 100,000 alive at age 10, 98,505 live for two years, and therefore the probability of living two years, or

$${}_2p_{10} = \frac{98,505}{100,000} = .98505$$

The probability of living two years may be viewed as the probability of two independent events both happening, viz.: (1) that (x) will first live one year, and (2) that $(x+1)$ will live a second year, and therefore

$${}_2p_x = p_x \times p_{x+1}$$

Of course this expression is the exact equivalent of $\frac{l_{x+2}}{l_x}$

because $$p_x \times p_{x+1} = \frac{l_{x+1}}{l_x} \times \frac{l_{x+2}}{l_{x+1}}$$

$$= \frac{l_{x+2}}{l_x}$$

The general expression for the probability of living any number of years is obtained in the same way as the foregoing, seeing that l_{x+n} persons are alive after n years out of l_x persons at age x. Therefore

$${}_np_x = \frac{l_{x+n}}{l_x} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(31)$$

Care must be taken not to confuse this expression, the proba-

bility of living n years, with p_{x+n} which is the probability of living for one year at age $x+n$, that is,

$$p_{x+n} = \frac{l_{x+n+1}}{l_{x+n}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (32)$$

The probabilities corresponding to the above, relating to the deaths instead of the numbers living, are as follows:—

$$q_x = \frac{d_x}{l_x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots . (33)$$

$$= \frac{l_x - l_{x+1}}{l_x}$$

$$= 1 - p_x \quad \text{as before.}$$

$$|_2 q_x = \frac{d_x + d_{x+1}}{l_x}$$

$$= \frac{l_x - l_{x+2}}{l_x} = 1 - {}_2p_x$$

$$|_n q_x = \frac{d_x + d_{x+1} + d_{x+2} + \ldots\ldots + d_{x+n-1}}{l_x}$$

$$= \frac{l_x - l_{x+n}}{l_x}$$

$$= 1 - {}_np_x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (34)$$

${}_{n-1}|q_x$ represents the probability of dying in the n^{th} year; it is therefore found by dividing the number of deaths which take place in the n^{th} year by the number alive at age x:—

$$_{n-1}|q_x = \frac{d_{x+n-1}}{l_x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (35)$$

$$= \frac{l_{x+n-1} - l_{x+n}}{l_x}$$

$$= {}_{n-1}p_x - {}_np_x \quad \ldots\ldots\ldots\ldots (36)$$

Mortality Table from Probabilities. In the practical formation of a mortality table, probably the most usual and effective process is to deduce the probabilities of living at each age, and, assuming a convenient radix for the commencement of the table, to multiply the values of l_x by the successive values of p_x, thus forming the entire table. If we have a

complete table of values of p_x we can form the values of the l_x column as follows: —

$$l_{x+1} = l_x \times p_x \quad \text{as above;}$$

where l_x is any number which it may be convenient to adopt as a radix. Having thus found l_{x+1} we obtain

$$l_{x+2} = l_{x+1} \times p_{x+1}$$
$$l_{x+3} = l_{x+2} \times p_{x+2}$$
and so forth.

The probabilities at individual ages can be formed from numbers which are quite independent of the numbers at other ages; and this is usually done in the investigation of the mortality of life assurance companies. For example, we may have 5,000 persons alive at the age of 30 out of whom 42 deaths might take place. Therefore the probability of living a year at age 30 would be .9916. At age 25, on the other hand, we might only have 4,000 alive amongst whom 32 deaths would occur, and therefore the probability of living a year at age 25 would work out as .9920; similarly for other ages, the actual numbers under observation at any age being quite independent of the numbers observed at other ages when an investigation is conducted in this manner.

Select Lives and Probabilities. It has already been said that the average mortality amongst persons who have been recently examined for life assurance, and who are therefore "select" lives, is much less than that experienced after the lapse of a few years when the lives are "mixed". It follows, therefore, that the probability of living a year on the part of a select life is greater than the same probability for mixed lives, and the probability of dying less. When representing select functions the age at which selection took place is placed in square brackets. Thus $p_{[x]}$ denotes the probability that a life selected at age x will live one year. Any addition outside of the square brackets shows the period elapsed since the date of selection, so that $p_{[x]+1}$ denotes the probability that a person who was selected at age x, but is now aged $x + 1$, will live a year. In like manner, $l_{[x]}$ denotes the number selected at age x, $l_{[x]+1}$ the

number out of $l_{[x]}$ who survive one year, and $l_{[x]+n}$ the number surviving n years.

The number of lives selected at age $x+n$ would be stated as $l_{[x+n]}$. The difference between $l_{[x]+n}$ and $l_{[x+n]}$ should be carefully noted, but the distinction is a clear one, and the reason for it readily understood. The effects of selection are not supposed to be of much importance after five years; but, during the first five years, there is a marked difference in the mortality rates of those of the same age who have been assured for different periods. Accordingly, in representing the numbers living amongst selected lives, six columns are necessary where only one was required for an ordinary mixed table. The following will illustrate the method of tabulation. It is a brief extract from the

BRITISH OFFICES ANNUITY EXPERIENCE.

AGE AT ENTRY.	YEARS ELAPSED SINCE DATE OF PURCHASE.						AGE ATTAINED.
	0	1	2	3	4	5 or more.	
x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$	$l_{[x]+4}$	l_{x+5}	$x+5$
20	100000	99738	99387	98923	98349	97691	25
21	99329	99064	98711	98244	97665	97004	26
22	98655	98388	98031	97561	96978	96312	27
23	97979	97709	97349	96874	96286	95615	28
24	97299	97026	96662	96182	95589	94911	29

A similar table of $d_{[x]}$, $d_{[x]+1}$, $d_{[x]+2}$, etc., is published and is of course convenient, although all these values may be obtained from the numbers living, because

$$d_{[x]} = l_{[x]} - l_{[x]+1} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (37)$$
$$d_{[x]+1} = l_{[x]+1} - l_{[x]+2}$$
$$d_{[x+n]} = l_{[x+n]} - l_{[x+n]+1}$$

Out of $l_{[x]}$ persons at age x, $l_{[x]+1}$ survive one year, and we have therefore the probability of living:—

$$p_{[x]} = \frac{l_{[x]+1}}{l_{[x]}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (38)$$

whence $p_{[x]} \times l_{[x]} = l_{[x]+1}$

The modifications on the other formulas for probabilities necessary to express the values in select functions can be easily obtained and proofs of the individual expressions are probably unnecessary. Some of the actual formulas may, however, be given for ready reference, as follows:—

$$p_{[x]+n} = \frac{l_{[x]+n+1}}{l_{[x]+n}} \qquad \text{(39)}$$

$$p_{[x+n]} = \frac{l_{[x+n]+1}}{l_{[x+n]}} \qquad \text{(40)}$$

$$_n p_{[x]} = \frac{l_{[x]+n}}{l_{[x]}} \qquad \text{(41)}$$

$$|_n q_{[x]} = \frac{l_{[x]} - l_{[x]+n}}{l_{[x]}} = 1 - {}_n p_{[x]} \qquad \text{(42)}$$

$$_{n-1}|q_{[x]} = \frac{d_{[x]+n-1}}{l_{[x]}} = {}_{n-1}p_{[x]} - {}_n p_{[x]} \qquad \text{(43)}$$

Joint Probabilities. The probability that two lives, (x) and (y), will both survive a year is the product of two independent probabilities, namely:—

$$p_x \times p_y = \frac{l_{x+1}}{l_x} \times \frac{l_{y+1}}{l_y}$$

or, as it is usually written $= \dfrac{l_{x+1:y+1}}{l_{xy}}$ ....(44)

l_{xy} represents the total number of pairs of persons which may be formed from l_x of the age x and l_y of the age y, because each person aged x may be paired off with each person aged y. Therefore each one of the l_x persons forms l_y pairs, and the entire number of l_x persons will therefore form $l_x \times l_y$ or l_{xy} pairs. If the first death be that of one of the l_x persons, it will cause the dissolution of l_y pairs. At the end of the year, when the numbers of persons would be reduced to l_{x+1} and l_{y+1} respectively, the total number of pairs which could then be formed would be $l_{x+1} \times l_{y+1} = l_{x+1:y+1}$, and therefore the probability that any one pair out of the original l_{xy} pairs would remain unbroken would be $\dfrac{l_{x+1:y+1}}{l_{xy}}$ as before.

There are certain other questions in probability involving two lives which may be of more or less interest.

The probability that (x) will die within n years, and that (y) will survive, is

$$|_n q_x \times {}_n p_y = (1 - {}_n p_x)\, {}_n p_y = {}_n p_y - {}_n p_{xy}$$

The probability that both will die within n years is

$$\begin{aligned} |_n q_{\overline{xy}} &= |_n q_x \times |_n q_y = (1 - {}_n p_x)(1 - {}_n p_y) \\ &= 1 - {}_n p_x - {}_n p_y + {}_n p_{xy} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (45) \end{aligned}$$

The probability that one or other will die within n years, but not both, is the sum of two probabilities, namely:—

1. That (x) will so die, and (y) survive; and
2. That (y) will so die, and (x) survive. Therefore the total probability sought is

$$\begin{aligned} |_n q_x \times {}_n p_y + |_n q_y \times {}_n p_x &= (1 - {}_n p_x)\, {}_n p_y + (1 - {}_n p_y)\, {}_n p_x \\ &= {}_n p_y - {}_n p_{xy} + {}_n p_x - {}_n p_{xy} \\ &= {}_n p_x + {}_n p_y - 2\, {}_n p_{xy} \end{aligned}$$

The probability that the last survivor of (x) and (y) will survive n years is composed of three separate probabilities, namely:

1. That (x) will survive, (y) having died;
2. That (y) will survive, (x) having died; and
3. That both (x) and (y) will survive.

The entire probability is therefore

$$\begin{aligned} {}_n p_{\overline{xy}} &= {}_n p_x \times |_n q_y + {}_n p_y \times |_n q_x + {}_n p_{xy} \\ &= {}_n p_x - {}_n p_{xy} + {}_n p_y - {}_n p_{xy} + {}_n p_{xy} \\ &= {}_n p_x + {}_n p_y - {}_n p_{xy} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (46) \end{aligned}$$

The same results may frequently be obtained in other ways, such as the probability that the last survivor (*i. e.*, both) will *die* within n years; this is evidently the complement of the probability that the last survivor *will live n years* and is therefore

$$|_n q_{\overline{xy}} = 1 - ({}_n p_x + {}_n p_y - {}_n p_{xy})$$

mathematically the same result as before.

Expectation of Life. The term "Expectation of Life" is one which is much used, and is frequently *supposed* by the uninstructed to be the basis of actuarial calculation. The function has often been *erroneously used* for calculations involving monetary payments, and the misunderstanding is one which it is exceedingly difficult to correct. Writer after writer on such subjects has called attention to the true situation, and yet the error survives of supposing that a calculation for a fixed period equal to the expectation of life will give an accurate result, the same as a calculation made correctly by using the probabilities of living each year during life. By taking the expectation of life and calculating an annuity for this period as a fixed term, the result is always greater than the true value of a life annuity. If a similar method were applied to the calculations of life assurance premiums the results would always be too small.

"Expectation of Life" means the average number of years which persons of a given age will survive. It has no relation whatever to the time when any individual is likely to die, nor to the time when death is most probable. The individual may die to-morrow or he may live for 50 years, and yet the expectation of life for his age may be a period like 20 or 25 years. To find the average number of years which persons of the age x will live, we may consider a group of l_x persons. At the end of the first year when they have attained age $x+1$ the number of survivors will be l_{x+1} and these persons will each have lived one year. At the end of the second year the number of persons alive will be l_{x+2} and these persons will each have lived for a second year, and so on, until the limit of life is reached, the number surviving any year, say the n^{th}, being l_{x+n}. The total number of complete years which the l_x persons would live in the aggregate would therefore be

$$l_{x+1} + l_{x+2} + l_{x+3} + \ldots\ldots\ldots\ldots$$

If this number of years be divided by the number of persons, l_x, in the original group, we have the average for each, which is called the "curtate" expectation of life, and is written e_x, namely:

$$e_x = \frac{l_{x+1} + l_{x+2} + l_{x+3} + \ldots\ldots}{l_x} \qquad \ldots\ldots\ldots\ldots(47)$$

Complete Expectation. The above formula neglects altogether that portion of a year lived by each person in the year of death. The deaths do not take place at the end of the year, but at intervals throughout the year. Accordingly, on the average, in the year of death each person will live half a year. The entire l_x persons will therefore live on the average $\frac{1}{2}\, l_x$ years more than is shown in the above formula. To obtain a "complete" expectation of life, therefore, we must add $\frac{1}{2}\, l_x$ to the numerator of the above fraction and we have

$$\overset{\circ}{e}_x = \frac{\frac{1}{2}\, l_x + l_{x+1} + l_{x+2} + l_{x+3} + \ldots\ldots}{l_x}$$

$$= \tfrac{1}{2} + e_x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (48)$$

As above indicated, many persons confuse the term "Expectation of Life" with the *most probable lifetime;* or they suppose that a person has an even chance of surviving his expectation. Neither supposition is correct.

Most Probable Lifetime. The "most probable lifetime" must be found by reference to the mortality table itself, by observing at what age the largest number of deaths occurs. The age thus found represents the age at which death is most likely to happen, and the difference between that age and the present age is therefore the *most probable lifetime.* For example, in infancy the most probable lifetime is frequently less than a single year. It varies however in different mortality tables, and after the period of infancy is passed, the most probable age at death in the various tables is as follows:—

In the Northampton Table it is any age from 51 to 61 (say age 56)
" " Carlisle " " " age ....74 to 75
" " American " " " " ....73 to 74
" " Actuaries' " " " " ....72 to 73
" " New British O^M " " " " ....73 to 74

At age 30, therefore, by the American Experience Table the most probable lifetime is 43 years, yet the complete expectation of life is only 35.3 years. At age 70 the most probable lifetime is 3 years, while the expectation of life is 8.5 years.

Vie Probable. The *Vie probable*, or "probable lifetime", is used to denote the number of years which a person has an even chance of living. It represents the difference between the age attained and the age when the number alive by the mortality table at that age will be reduced exactly to one-half. In other words, if 10,000 persons are alive at age x the number of years which a person of that age has an even chance of living is represented by the difference between his present age and the age when only 5,000 out of the 10,000 persons still survive. At age 30 by the American Experience Table the *Vie probable* is nearly 38 years, and at age 70 it is nearly 8 years.

CHAPTER VII.

Life Annuities.

Contract Consideration and Benefit.

The two mathematical factors which enter into the computation of life assurance benefits are the interest on money earned by the assurance company and the mortality affecting those who take policies. These two factors have been investigated separately in Chapters V. and VI. Now we may combine them and consider actual contracts. An Ordinary Whole Life Policy may be divided in two parts. The first is the *consideration* which the applicant has to pay to obtain the policy. That consideration takes the form of a yearly payment to be made throughout his life; it is therefore a life annuity which he has to pay. In consideration of this payment he obtains the *benefit* of assurance protection during his life, the sum assured to be payable at death. These two parts of the assurance policy may therefore be analyzed separately; together they form the complete contract between the applicant and the company.

There are two methods of examining questions in life contingencies, both of which are correct, and which therefore bring out the same results. The one is to deal solely with the probabilities of living or dying and combine with these the interest on money: this method may be applied to the individual. The other method takes into consideration a group of persons, follows them year by year by the mortality table as they increase in age, and divides the payment made by, or to, the whole group by the number forming the group, thereby obtaining the average payment for each person. Sometimes one method will be used and sometimes the other, but it should be clearly understood that either method would produce the same result.

Interest and Probability Combined. We have already seen that the probability of living a year at age x, represented by p_x, is $\frac{l_{x+1}}{l_x}$. If an obligation be incurred to pay 1 to any person of the age x at the end of a year on condition that he be then alive, the present value of this obligation will be v, the present value of 1 due a year hence, multiplied by the probability of the payment being made; the entire present value is therefore vp_x. This general result may also be obtained by the group method. If the obligation be incurred in the case of all the l_x persons, then at the end of the year the total sum payable will be l_{x+1}, being 1 for each of the persons who survive. The present value at the beginning of the year will therefore be vl_{x+1}; and, if this present value be divided amongst the l_x persons alive at the beginning of the year, we have the share of each:—

$$\frac{vl_{x+1}}{l_x} = vp_x, \text{ the same as before.}$$

Pure Endowment. In like manner the obligation may guarantee the payment of 1 to a person provided he be alive at the end of two years, and the value of this will be $v^2 \, {}_2p_x$; or, assuming again that the obligation be entered into for each of the l_x persons, then the survivors being l_{x+2}, the total present value will be $v^2 \, l_{x+2}$ and the share of each

$$\frac{v^2 l_{x+2}}{l_x} = v^2 \, {}_2p_x, \text{ as before.}$$

If the obligation be incurred for the end of n years, then as there will be l_{x+n} survivors out of l_x persons, the net payment required to secure 1 to each of the survivors will be $v^n \, l_{x+n}$, and the payment from each will be $\frac{v^n \, l_{x+n}}{l_x}$. This is called a *Pure Endowment*, the single premium for which is represented by the symbol ${}_nE_x$, and therefore

$${}_nE_x = \frac{v^n \, l_{x+n}}{l_x} = v^n \, {}_np_x \qquad \ldots\ldots\ldots\ldots(49)$$

Life Annuity. A Life Annuity is simply a series of Pure Endowments payable at the end of each year, provided the annuitant be then alive. It is represented by the symbol a_x and we have therefore

$$a_x = vp_x + v^2\ {}_2p_x + v^3\ {}_3p_x + \dots\dots \qquad (50)$$

Or, considering again the purchase of a life annuity by each one of the l_x persons, we would have for the present value of all the payments to be made

$$vl_{x+1} + v^2\, l_{x+2} + v^3\, l_{x+3} + \dots\dots\dots\dots$$

and the present value in respect of each of the l_x persons would therefore be

$$a_x = \frac{v\, l_{x+1} + v^2\, l_{x+2} + v^3\, l_{x+3} + \dots\dots\dots}{l_x} \qquad (51)$$

which is another way of expressing the same formula as that given above. This formula may be transposed in an interesting and instructive manner. Multiplying top and bottom of the right-hand expression by l_{x+1}, and taking v outside the bracket in the numerator, we have

$$a_x = \frac{vl_{x+1}\,(l_{x+1} + vl_{x+2} + v^2l_{x+3} + \dots)}{l_x \times l_{x+1}}$$

And as $\frac{l_{x+1}}{l_x} = p_x$, we may write this expression

$$a_x = vp_x \left(1 + \frac{v\, l_{x+2} + v^2\, l_{x+3} + \dots\dots\dots\dots\dots}{l_{x+1}}\right)$$

$$= vp_x\,(1 + a_{x+1})$$

Whence the value of an annuity at age x can be obtained from one at age $x+1$ if we know the probability of living at age x and the rate of interest.

These are the simplest benefits combining compound interest with the probabilities of life, and they are easily understood.

Annuity Due. When the first payment under an annuity is made at once, as in the case of premiums on a policy of assurance, the symbol is expressed as $\mathbf{a}_x$, and it includes the immediate payment in addition to the usual future payments provided by a_x. We therefore have

$$\mathbf{a}_x = 1 + a_x \qquad \dots\dots\dots\dots\dots\dots (52)$$

Temporary Annuity. But there are many life assurance benefits which do not run for the whole duration of life; and it is necessary to investigate the values of temporary annuities. We previously found the value of an annuity-certain for n years represented by the symbol $a_{\overline{n}|}$. If the temporary annuity depend upon a life (x), to terminate certainly after n years but sooner if (x) should die, the symbol used is $a_{x\overline{n}|}$, and we can find the value from fundamental principles in the same way as the value of a life annuity was deduced. If l_x persons all purchase annuities on their lives of 1 each to run for n years to those who live so long, then the present values of the several payments will be

$$v\,l_{x+1} + v^2\,l_{x+2} + v^3\,l_{x+3} + \ldots\ldots\ldots + v^n\,l_{x+n}$$

The total value of these payments divided by l_x will give the value of a temporary annuity, namely:—

$$a_{x\overline{n}|} = \frac{vl_{x+1} + v^2 l_{x+2} + \ldots\ldots\ldots + v^n l_{x+n}}{l_x} \qquad \ldots\ldots(53)$$

Deferred Annuity. A *deferred annuity* is one which commences only after the lapse of a number of years, payable if (x) be then alive and only so long thereafter as (x) may live. In such case the annuity is "entered upon" after n years, so that *the first payment would take place in* ($n+1$) *years*. If l_x such annuities were granted, then, as l_{x+n+1} persons would be alive when the first payments fall due, l_{x+n+2} when the second payments fall due, etc., the present value of all these payments would be

$$v^{n+1}\,l_{x+n+1} + v^{n+2}\,l_{x+n+2} + \ldots\ldots\ldots$$

and this total divided amongst each of the l_x persons would be the present value of the deferred annuity, viz:—

$$_{n|}a_x = \frac{v^{n+1}\,l_{x+n+1} + v^{n+2}\,l_{x+n+2} + \ldots\ldots}{l_x} \qquad (54)$$

Now it will be observed that the temporary annuity consists of those payments under a life annuity which are made in the early years; while the deferred annuity consists of those which are made later, and the two combined, namely,

$$\frac{v\,l_{x+1} + v^2\,l_{x+2} + \ldots\ldots + v^n\,l_{x+n}}{l_x} + \frac{v^{n+1}\,l_{x+n+1} + v^{n+2}\,l_{x+n+2}\ldots\ldots}{l_x}$$

make up the annuity for the whole of life as one would naturally expect; and therefore we have

$$a_x = a_{x\overline{n}|} + {}_{n|}a_x \qquad \ldots\ldots\ldots\ldots\ldots\ldots(55)$$

whence $$a_{x\overline{n}|} = a_x - {}_{n|}a_x \qquad \ldots\ldots\ldots\ldots\ldots\ldots(56)$$

and $${}_{n|}a_x = a_x - a_{x\overline{n}|} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(57)$$

We may modify the deferred annuity in an interesting way and obtain another useful formula:—

$${}_{n|}a_x = \frac{v^{n+1}\,l_{x+n+1} + v^{n+2}\,l_{x+n+2} + \ldots\ldots}{l_x}$$

Multiply top and bottom of the right side of this expression by l_{x+n} and take out v^n as a common factor, and we have

$$\begin{aligned} {}_{n|}a_x &= \frac{v^n\,l_{x+n}}{l_{x+n}} \times \frac{v\,l_{x+n+1} + v^2\,l_{x+n+2} + \ldots\ldots}{l_x} \\ &= \frac{v^n\,l_{x+n}}{l_x} \times \frac{v\,l_{x+n+1} + v^2\,l_{x+n+2} + \ldots\ldots}{l_{x+n}} \\ &= v^n\,{}_np_x\,a_{x+n} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (58) \end{aligned}$$

This shows that the deferred annuity on (x) is equal to a life annuity on $(x+n)$ discounted to the present time at interest and multiplied by the probability of (x) surviving n years. This result might have been obtained by direct reasoning without mathematical demonstration.

m times a Year. All the foregoing annuity formulas are based upon payments taking place once a year, but frequently arrangements are made whereby annuity payments and premiums fall due at semi-annual or quarterly intervals: under industrial policies, indeed, the premiums are collected weekly. It becomes necessary, therefore, to investigate the value of an annuity payable by semi-annual, quarterly, and other instalments. Dealing with the general form of an annuity payable m times a year, the

symbol for this annuity is written $a_x^{(m)}$. We can obtain an approximate value as follows:—

When the first payment under a life annuity is due at once, its value is $\mathbf{a}_x$, which may be written............ $_0|\mathbf{a}_x = 1 + a_x$
The value of an annuity deferred one year is... $_1|\mathbf{a}_x = 0 + a_x$
The value of an annuity deferred one m^{th} part of a year would therefore be approximately in proportion, viz.:—

$$_{\frac{1}{m}}|\mathbf{a}_x = \frac{m-1}{m} + a_x$$

and similarly

$$_{\frac{2}{m}}|\mathbf{a}_x = \frac{m-2}{m} + a_x$$

Suppose that we have m annuities in all, the first payments under which fall due at the end of $\frac{1}{m}, \frac{2}{m}, \frac{3}{m}$, etc., of a year respectively, then the entire series of annuities will provide 1 at the end of each m^{th} part of a year, and the complete series of values will be m times the value of an annuity of 1 payable in instalments of $\frac{1}{m}$. The complete series of values will be as follows:—

$$ma_x^{(m)} = a_x + \frac{m-1}{m} + a_x + \frac{m-2}{m} + a_x + \frac{m-3}{m} + \ldots + a_x + \frac{m-m}{m}$$

$$= m\,a_x + \frac{m\,(m-1)}{2\,m}$$

from which we obtain an approximate formula for the value of an annuity payable m times a year, namely,

$$a_x^{(m)} = a_x + \frac{m-1}{2\,m} \qquad \text{..................(59)}$$

This approximation, although inaccurate in several minor particulars, is generally used in practice. From it we obtain values of annuities payable by half-yearly and quarterly instalments, namely,

$$a_x^{(2)} = a_x + \tfrac{1}{4} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(60)$$

$$a_x^{(4)} = a_x + \tfrac{3}{8} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(61)$$

and these values are sufficiently close for general purposes.

COMMUTATION COLUMNS.

The formulas for annuity values so far given seem to involve a great deal of arithmetical labor to obtain practical results; but this may be minimized in various ways—particularly by using commutation columns. In the expression for the value of a life annuity—

$$a_x = \frac{v\, l_{x+1} + v^2\, l_{x+2} + \ldots\ldots\ldots\ldots\ldots}{l_x}$$

it will be observed that each term would have to be calculated separately. The value of an annuity at age $x+1$ would be

$$a_{x+1} = \frac{v\, l_{x+2} + v^2\, l_{x+3} + v^3\, l_{x+4} + \ldots\ldots\ldots}{l_{x+1}}$$

On comparing the annuity at $x+1$ with the annuity at x it will be observed that no two of the terms are alike. Therefore on first consideration it would appear as if each term of the annuity at age $x+1$ would also have to be computed. This would involve a great amount of arithmetic in the calculation of annuity values, but luckily a simple device avoids much of this trouble. If we multiply the numerator and denominator of the annuity on (x) by v^x we have the index of v in each case equal to the age of the life. In like manner, if we multiply the numerator and denominator of the annuity at age $x+1$ by v^{x+1}, we find that the same condition holds and the two expressions become

$$a_x = \frac{v^{x+1}\, l_{x+1} + v^{x+2}\, l_{x+2} + v^{x+3}\, l_{x+3} + \ldots}{v^x\, l_x}$$

$$\text{and } a_{x+1} = \frac{v^{x+2}\, l_{x+2} + v^{x+3}\, l_{x+3} + \ldots\ldots\ldots\ldots}{v^{x+1}\, l_{x+1}}$$

It will be observed that now the annuity a_x contains all the values included in the annuity a_{x+1}, so that when the values have been

thus calculated for the youngest age in the table, they are available for all other ages. For this function combining interest with the number living the symbol D is used, and we have

$$D_x = v^x l_x$$
$$D_{x+1} = v^{x+1} l_{x+1}$$
and, generally, $$D_{x+n} = v^{x+n} l_{x+n} \quad \text{.............}(62)$$

The expression for the annuity now becomes

$$a_x = \frac{D_{x+1} + D_{x+2} + D_{x+3} + \ldots\ldots\ldots}{D_x}$$

The sum of the values $D_{x+1} + D_{x+2} + D_{x+3} + \ldots\ldots\ldots\ldots$ is expressed as equal to N_x when the value of the annuity becomes in symbols

$$a_x = \frac{N_x}{D_x} \quad \text{...........................}(63)$$

This value shows the origin of the use of the letters N and D: they were used as being the initial letters of the words "Numerator" and "Denominator" in the expression for the value of an annuity.

The formation of such columns will be understood more readily by a practical example taken from the older ages of the

AMERICAN EXPERIENCE TABLE $3\frac{1}{2}\%$.

AGE. x	$D_x = v^x l_x$	$N_x = D_{x+1} + D_{x+2} +$ &c.
85	29.4610	61.3342
86	21.7598	39.5744
87	15.4383	24.1361
88	10.3963	13.7398
89	6.5623	7.1775
90	3.8305	3.3470
91	2.0187	1.3283
92	.9119	.4164
93	.3222	.0942
94	.0828	.0114
95	.0114	.....

The use of the N column in the manner above explained where

$$N_x = D_{x+1} + D_{x+2} + D_{x+3} + \ldots\ldots \quad (64)$$

is that adopted by the International Congress of Actuaries, although it is different from that which has been generally employed in America. The practice in America has been to sum the D_x column from x upwards. This is really the more logical and satisfactory course to pursue, because all the other columns are summed in that way, and no advantage is gained by the method above outlined in the way of convenience. When the values are summed from D_x upwards, the N is now distinguished in America by using "full faced" type, as follows:—

$$\mathbf{N}_x = D_x + D_{x+1} + D_{x+2} + \ldots\ldots\ldots\ldots\ldots\ldots$$

In the past, however, the same type has been used for the one form as for the other, and great care must therefore be exercised to avoid mistake.

Select Forms. Occasionally, and especially in the case of select life functions, the values have been summed from x upwards and the form of letter expressing the summation given thus: N, the distinctive feature of this letter being the double bar in the centre, so that we have

$$N_{[x]} = D_{[x]} + D_{[x]+1} + D_{[x]+2} + \ldots\ldots\ldots \quad (65)$$

This method is usually adopted in Great Britain for distinguishing what is sometimes called the "initial" form, where the summation is from x upwards, from the "terminal" form, where the summation is from $x + 1$ upwards.

Functions of this nature are called "Commutation Columns", because they facilitate the commuting or interchanging of values at one age into values at another.

All of the annuity problems already discussed can be solved by the use of commutation columns, and the expressions derived in that form take very simple shapes which commend themselves to the reason and fix themselves readily in the mind. We have seen that an annuity for the whole of life can be represented by

$a_x = \dfrac{N_x}{D_x}$. In the case of a temporary annuity it will be seen on reference to formula (53) above, that on multiplying the top and bottom of the expression by v^x we have

$$a_{x\overline{n}|} = \frac{v^{x+1}\, l_{x+1} + v^{x+2}\, l_{x+2} + \ldots\ldots + v^{x+n}\ l_{x+n}}{v^x\, l_x}$$

$$= \frac{D_{x+1} + D_{x+2} + \ldots\ldots + D_{x+n}}{D_x}$$

which is evidently

$$a_{x\overline{n}|} = \frac{N_x - N_{x+n}}{D_x} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(66)$$

In like manner, the deferred annuity is the complement of the temporary annuity, namely,

$$_n|\,a_x = \frac{N_{x+n}}{D_x} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(67)$$

Still another form of annuity may be given although it is of very little practical use, namely, the "intercepted" or deferred-temporary annuity. $_n|_m\,a_x$ represents an annuity on (x) to commence after n years and then to run only for m years—in other words, it runs from age $x + n$ to age $x + n + m$ if (x) should be alive until the latter age, and we have

$$_n|_m\,a_x = \frac{N_{x+n} - N_{x+n+m}}{D_x} \qquad \ldots\ldots\ldots\ (68)$$

In commutation symbols the Pure Endowment is expressed

$$_nE_x = \frac{D_{x+n}}{D_x} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (69)$$

which can be easily proved.

In the case of an annuity-due, as the first payment would be made at once, then in commutation symbols it would be expressed as

$$\mathbf{a}_x = 1 + a_x = 1 + \frac{N_x}{D_x} = \frac{D_x + N_x}{D_x} = \frac{N_{x-1}}{D_x} \qquad \ldots\ldots (70)$$

As above mentioned, the use of commutation symbols produces expressions which are easy to remember and simple in their ap-

pearance. Nevertheless it is seldom necessary in practical work to use these functions for the calculation of benefits, because for all ordinary tables the annuity values themselves are available. In computing tables of the annuity values, however, it is usual and convenient to form commutation columns in the first instance, and thereafter derive the final functions. Sometimes, also, when tables of temporary annuities are not available, the commutation columns afford an easy means of obtaining the required values.

CHAPTER VIII.

NET PREMIUMS.

The Net Premium is the mathematical equivalent of the benefit, according to the mortality table and rate of interest employed in the calculation. It is somewhat of the nature of the manufacturing cost of goods; and must be distinguished from the selling price or premium actually charged, which will be hereafter treated. In the last chapter annuities were discussed to an extent sufficient to explain the nature of the consideration for a life policy. We shall now proceed to obtain expressions for the value of the benefit, or single premium, and thereafter by equating the two find the net annual premiums.

Single Premium. The benefit under a Whole Life Policy will be best understood by considering that each one of the l_x persons forming the group at age x effects a policy for 1 on his life. If this be done the number of deaths in the first year would be d_x, and the value at the beginning of the year of the claims by death due at the end would be vd_x; the deaths in the second year would be d_{x+1}, and the value of the claims at the beginning of the first year would be $v^2 d_{x+1}$; similarly for subsequent years. The total value of all the death claims would therefore be

$$v\, d_x + v^2\, d_{x+1} + v^3\, d_{x+2} + \&c.$$

If this present value be distributed amongst the l_x persons we shall have as the share of each, the value of the benefit of the assurance of 1 at death. In technical language, we thus find the net single premium for a Whole Life assurance of 1, which is represented by A_x, A being the initial letter of the word "Assurance":—

$$A_x = \frac{vd_x + v^2 d_{x+1} + v^3 d_{x+2} + v^4 d_{x+3} + \&c.}{l_x} \tag{71}$$

This is the value of the benefit to the life assured, represented by the payment of 1 at his death, and it therefore represents the smallest single premium which can be accepted by a company as the consideration for its undertaking such an obligation.

The above formula may be transformed in an instructive manner as follows:—

NB

$$\begin{aligned} A_x &= \frac{v\,d_x + v^2 d_{x+1} + v^3 d_{x+2} + \dots\dots\dots}{l_x} \\ &= \frac{v\,(l_x - l_{x+1}) + v^2\,(l_{x+1} - l_{x+2}) + v^3\,(l_{x+2} - l_{x+3}) + \dots}{l_x} \\ &= \frac{v\,l_x + v^2\,l_{x+1} + v^3\,l_{x+2}\,\dots}{l_x} - \frac{v\,l_{x+1} + v^2\,l_{x+2} + \dots\dots}{l_x} \\ &= v\,(1 + \frac{v\,l_{x+1} + v^2\,l_{x+2}\,\dots\dots\dots}{l_x}) - a_x \\ &= v\,(1 + a_x) - a_x \quad \dots\dots\dots\dots\dots\dots \end{aligned} \tag{72}$$

Assurance Relationship to Annuity.

This shows that the value of the sum assured payable at death may be conveniently expressed in terms of an annuity payable during life, and the rate of interest. This formula again may be mathematically transformed so as to produce other expressions giving the assurance value in terms of the annuity:—

$$\begin{aligned} A_x &= v\,(1 + a_x) - a_x \\ &= \frac{1 + a_x}{1 + i} - a_x = \frac{1 + \cancel{a_x} - \cancel{a_x} - ia_x}{1 + i} \\ &= \frac{1 - ia_x}{1 + i} \quad \dots\dots\dots\dots\dots\dots \quad (73) \\ &= v - da_x \\ &= 1 - d - da_x \\ &= 1 - d\,(1 + a_x) \quad \dots\dots\dots\dots \quad (74) \end{aligned}$$

NB

This last formula is one of great importance, and may be explained verbally so as to assist the memory. The sum assured under the policy is 1, and if it were payable at once its value would of course be 1; but, as it is not payable until the end of the year of death, it follows that interest in advance for each year

(d each year) must be deducted during the lifetime of (x). This interest is of the nature of an annuity-due, and its capitalized value is represented by the expression $d\ (1+a_x)$, which, deducted from the full sum of 1 gives the present value of the payment due at the end of the year of death.

Annual Premiums. Policies of life assurance are very seldom effected by single premium: the more common form of policy is that by continued annual payments during life. In whatever form, however, the payments are received by the assurance company, the net equivalent value must be the same. If an annual premium be due at the beginning of each year, it is, as formerly explained, an annuity-due payable by the policy-holder to the company in consideration of the obligation the company incurs. If such net annual premium be represented by P_x, the value of the total premiums throughout life would be $P_x\ (1+a_x)$, and this value must be the equivalent of the obligation incurred, namely A_x. We therefore have the formula:—

$$P_x\ (1+a_x) = A_x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (75)$$

whence the value of P_x may be derived:—

$$P_x = \frac{A_x}{1+a_x} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (76)$$

The general principle above indicated, that the value of the benefit must equal the value of the payment, is an important one, and is the fundamental principle in old-line life assurance.

It has been shown that the single premium may be expressed in terms of the annuity value and the rate of interest. By transformation the annuity value may be expressed in terms of the single premium and the rate of interest. For example, making use of formula (73) above—

$$A_x = \frac{1 - ia_x}{1+i}$$

we have

$$(1+i)\ A_x = 1 - ia_x$$

$$\therefore\ ia_x = 1 - (1+i)\ A_x$$

and

$$a_x = \frac{1-(1+i)\ A_x}{i} \quad \ldots\ldots\ldots \quad (77)$$

The annual premium P_x may also be expressed in terms of annuity values or of single premiums, because we have

$$P_x = \frac{A_x}{1 + a_x} = \frac{1 - d\,(1 + a_x)}{1 + a_x}$$

$$= \frac{1}{1 + a_x} - d; \qquad \qquad (78)$$

Or again

$$P_x = \frac{A_x}{1 + \dfrac{1 - (1+i)\,A_x}{i}}$$

$$= \frac{i\,A_x}{(1+i) - (1+i)\,A_x} \qquad \qquad (79)$$

Dividing numerator and denominator of the right side by $(1+i)$, we place this in a much neater form:—

$$P_x = \frac{d\,A_x}{1 - A_x} \qquad \qquad (80)$$

And lastly, we may express the annuity value in terms of the premium from equation (78) above:—

$$P_x = \frac{1}{1 + a_x} - d$$

$$P_x + d = \frac{1}{1 + a_x}$$

$$(1 + a_x)\,(P_x + d) = 1$$

$$1 + a_x = \frac{1}{P_x + d}$$

and

$$a_x = \frac{1}{P_x + d} - 1 \qquad \qquad (81)$$

It will be seen, therefore, that these three functions, the annuity, the single premium, and the annual premium, are all closely related to one another. If we know any one and the rate of interest used for the calculation, we can derive the other two. It is natural that they should be so related, because all three benefits depend solely on the duration of human life, and terminate or mature, as the case may be, in event of death. Advantage has been taken of the close relationship between annuities and premiums to compute tables from which, if the value of

one be known, the value of the other may be at once found by simple reference to the table.

A verbal explanation could be given for each of the above formulas, and students should satisfy themselves not only of the mathematical accuracy of the expressions, but also of their reasonableness.

Temporary and Deferred Assurance. In a manner similar to that used for the temporary annuity, the value of a temporary assurance may be found, namely:—

$$A^{1}_{x\,\overline{n}|} = \frac{vd_x + v^2 d_{x+1} + \ldots\ldots + v^n d_{x+n-1}}{l_x} \qquad \ldots\ldots\ldots\ldots(82)$$

The symbol for this may be written either as $A^{1}_{x\,\overline{n}|}$; or, following the principle applied to annuities, as $|_n A_x$. The single premium for a Pure Endowment was expressed in formula (49) as ${}_nE_x$; but it may likewise be represented by $A_{x\,\overline{n}|}^{\;\;1}$ where the 1 above $\overline{n}|$ indicates that the sum assured is payable only if that term should first expire; in other words, only if (x) be alive after n years. In the case of the term assurance the 1 is above the x, indicating that (x) must die within the n years in order that the sum assured shall be payable.

The deferred assurance is little used in practical transactions, but it may be given here for the sake of completeness, namely:—

$$
\begin{aligned}
{}_{n|}A_x &= \frac{v^{n+1} d_{x+n} + v^{n+2} d_{x+n+1} + \ldots\ldots\ldots\ldots\ldots}{l_x} \\
&= \frac{v^n l_{x+n}}{l_x} \times \frac{v\, d_{x+n} + v^2 d_{x+n+1} + \ldots\ldots\ldots}{l_{x+n}} \\
&= v^n\, {}_np_x\, A_{x+n} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (83)
\end{aligned}
$$

Endowment Assurance. An Endowment Assurance provides for the payment of the sum assured either at death within n years or on survivance, and it consists, therefore, of the temporary assurance above mentioned combined with a pure endowment to provide the amount in event of survivance. These

two benefits are mutually exclusive, because, if the sum assured become payable under the term portion, the endowment is not payable; while, on the other hand, if the endowment be payable on survivance, the term assurance would have expired in normal course. The combination of the two gives the single premium for an endowment assurance, namely:—

$$A_{x\overline{n}|} = A^1_{x\overline{n}|} + {}_nE_x \quad \ldots\ldots\ldots\ldots\ldots\ldots (84)$$

To express the temporary assurance and the endowment assurance in terms of annuity values, let us first investigate formula (82) for the value of a temporary assurance:—

$$A^1_{x\overline{n}|} = \frac{vd_x + v^2d_{x+1} + v^3d_{x+2} + \ldots\ldots + v^nd_{x+n-1}}{l_x}$$

$$= \frac{v(l_x - l_{x+1}) + v^2(l_{x+1} - l_{x+2}) + \ldots\ldots + v^n(l_{x+n-1} - l_{x+n})}{l_x}$$

$$= \frac{vl_x + v^2l_{x+1} + \ldots + v^nl_{x+n-1}}{l_x} - \frac{vl_{x+1} + v^2l_{x+2} + \ldots v^nl_{x+n}}{l_x}$$

$$= v(1 + a_{x\overline{n-1}|}) - a_{x\overline{n}|} \quad \ldots\ldots\ldots\ldots\ldots\ldots (85)$$

Now if we add to the above expression the value of a pure endowment maturing in n years, namely, $\frac{v^n l_{x+n}}{l_x}$, we shall only cancel the last term of the second annuity, that is

$$-a_{x\overline{n}|} + \frac{v^n l_{x+n}}{l_x} = -a_{x\overline{n-1}|}$$

We therefore have

$$A_{x\overline{n}|} = v(1 + a_{x\overline{n-1}|}) - a_{x\overline{n-1}|} \quad \ldots\ldots\ldots\ldots\ldots\ldots (86)$$

which is obviously correct, because the first expression on the right side of this equation provides 1 at the end of each year on which a person enters for n years, and the second expression subtracts 1 for each year a person survives, but only for $(n-1)$ years. Accordingly, if he should enter on any year and not survive it, the sum of 1 would be free for payment under the temporary assurance, while if he should survive the last year, the sum of 1 would likewise be free, because there is no deduction under the second annuity for that year.

From the last equation we can obtain another expression which is more useful, as follows:—

$$A_{x\,\overline{n}|} = v\,(1 + a_{x\,\overline{n-1}|}) - a_{x\,\overline{n-1}|}$$

$$= v + v\,a_{x\,\overline{n-1}|} - a_{x\,\overline{n-1}|}$$

$$= 1 - d - (1 - v)\,a_{x\,\overline{n-1}|}$$

$$= 1 - d\,(1 + a_{x\,\overline{n-1}|}) \qquad \text{.................} (87)$$

The form of this expression is similar to that of formula (74), and the same kind of verbal reasoning may be applied to it.

Other Annual Premiums. We may now pass to the computation of annual premiums for limited payment policies, term assurances, etc. If a Whole Life policy be taken by limited annual premiums over n years, the value of the sum assured payable at death remains the same, but the premiums fall due over a shorter period; therefore the premium during such shorter period must be larger so that the aggregate payments may equal the benefit. In such circumstances the net premium is represented by the symbol ${}_nP_x$, and the total value of the net premiums is found by multiplying the annual payment by a temporary annuity for n years. The equation is:—

$${}_nP_x\,(1 + a_{x\,\overline{n-1}|}) = A_x$$

From this the value of the premium may be deduced, namely:—

$${}_nP_x = \frac{A_x}{1 + a_{x\,\overline{n-1}|}} \qquad (88)$$

The method of obtaining annual premiums for other classes of policies is similar. The annual premium for a temporary assurance on (x) is obtained from the following equation:—

$$P^{1}_{x\,\overline{n}|}(1 + a_{x\,\overline{n-1}|}) = A^{1}_{x\,\overline{n}|}$$

whence $$P^{1}_{x\,\overline{n}|} = \frac{A^{1}_{x\,\overline{n}|}}{1 + a_{x\,\overline{n-1}|}} \qquad (89)$$

In like manner the premium for a pure endowment payable on the expiry of n years, provided (x) be then alive, is

$$P_{x\overline{n|}}^{\;1} = \frac{A_{x\overline{n|}}^{\;1}}{1+a_{x\,\overline{n-1|}}}, \text{ or } \frac{v^n\,{}_np_x}{1+a_{x\,\overline{n-1|}}} \qquad (90)$$

Lastly, the annual premium for an Endowment Assurance payable in n years or at previous death is

$$P_{x\overline{n|}} = \frac{A_{x\overline{n|}}}{1+a_{x\,\overline{n-1|}}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (91)$$

Semi-Annual and Quarterly. The premiums thus far dealt with have been on a yearly basis, but frequently they are payable semi-annually or quarterly. In such cases the annuity value must be suitably adjusted in terms of formulas (60) and (61). $P_x^{(2)}$ represents the total amount each year when providing for two semi-annual payments; that is, the premium each half year would be $\frac{1}{2}\,P_x^{(2)}$. Therefore we have:—

$$P_x^{(2)}\,(\tfrac{1}{2} + a_x^{(2)}) = A_x$$

$$\text{whence}\quad P_x^{(2)} = \frac{A_x}{\frac{3}{4} + a_x} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (92)$$

and for a policy by quarterly premiums the formula would be

$$P_x^{(4)} = \frac{A_x}{\frac{5}{8} + a_x} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (93)$$

These results show that double the semi-annual premium will exceed the normal annual premium P_x, and likewise that four times the quarterly is greater than twice the semi-annual premium. As the result of actual calculation, it is found that the theoretic addition necessary to the annual premium in obtaining semi-annual payments is about 2%, and for quarterly payments about 3%. The theoretic addition only takes account of two elements:—(1) the possibility that part of the premium may be lost in the year of death; and (2) loss of interest which takes place through delay in payment of part of the premium. On an annual basis the premium would be paid in full at the beginning of the year of death,

whereas on a semi-annual or quarterly basis part of the premium might remain unpaid at the date of death; and interest on so much of the premium as is not paid at the beginning of the year is lost annually. In practice there is a third influence which is of considerable importance, namely, the additional expense incurred in collecting premiums twice or four times a year instead of once only.

General Practice. The general rule in America is to add 4% to the annual rate and thereafter divide by 2 to obtain the semi-annual premium; and to add 6% to the annual rate and thereafter divide by 4 to obtain the quarterly premium. These additions it will be observed are larger than are in theory required for true semi-annual and quarterly premiums if we leave out of account the additional expense. Nevertheless in practice the common custom is to use these semi-annual or quarterly payments as being *instalments of annual premiums*, and provide that in event of death before the full year's premium had been paid, any semi-annual or quarterly instalment then unpaid should be deducted from the sum assured at settlement. By these means a sufficient provision is made for the additional expense incurred under such policies.

Sum Assured Payable Immediately. The theory of assurances so far investigated has proceeded on the assumption that sums assured are payable at the end of the year in which death takes place. In this connection let us consider the simplest case of a term assurance for one year where

$$A^{1}_{x\overline{1|}} = \frac{vd_x}{l_x}$$

It will be seen that the claims d_x are discounted at interest for a complete year, and therefore the single premium provides that the sum assured be payable only at the end of the year. It was at one time the custom to pay sums assured in this way, that is, six months after death. But with the tendency towards broadening the view of life assurance, the practice has arisen of paying

claims immediately after death, or at least so soon as the proofs can be completed and the title of the claimant shown to be satisfactory. If sums assured are considered as payable at the moment of death, and if it be assumed that deaths on the average take place at equal intervals over the year, then it is apparent that, as compared with the theory so far investigated, the practical result of immediate payment after death will be a loss to the company of six months' interest on the average. To provide for payment of claims at death the above stated formula would become

$$\bar{A}^{1}_{x\overline{1|}} = \frac{v^{1/2}\, d_x}{l_x} = \frac{v d_x}{l_x}(1+i)^{1/2}$$
$$= (1+i)^{1/2} A^{1}_{x\overline{1|}}$$

When the sum assured is payable at the moment of death the assurance symbol has a bar placed above it as shown. The single premium for an ordinary assurance payable at the moment of death would become

$$\bar{A}_x = (1+i)^{1/2} A_x \quad \ldots\ldots\ldots\ldots\ldots\ldots(94)$$

because in every case provision has to be made for half a year's interest. The annual premium would be similarly modified:—

$$^{(\infty)}P_x{}^{*} = (1+i)^{1/2} P_x \quad \ldots\ldots\ldots\ldots\ldots\ldots(95)$$

The situation is different, however, in the case of Endowment Assurances, because the only portion of the benefit affected by the early payment of claims is the assurance portion. The endowment portion is not affected by the payment of death claims immediately when they arise, because the formula already provides that the Endowment shall be definitely paid at the end of n years. We have seen that

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* The use of $^{(\infty)}P_x$ appears to be a slight departure from the system of notation, as explained in the Appendix, but it is necessary because $\bar{P}_x$ represents the premium payable at infinitesimal intervals, whereas $^{(\infty)}P_x$ expresses the premium payable annually for an assurance due at the moment of death.

$$A_{x\overline{n}|} = A^{1}_{x\overline{n}|} + A_{x\overline{n}|}^{\;\;\,1}$$

and accordingly $\overline{A}_{x\overline{n}|} = (1+i)^{\frac{1}{2}} A^{1}_{x\overline{n}|} + A_{x\overline{n}|}^{\;\;\,1}$ ......(96)

As a matter of fact, claims cannot be paid actually at the moment of death. There is always more or less delay in submitting proofs, and allowance for this should be made. *For example*, if the average lapse of time be 1 month after death, then the loss of interest is for 5 months and the value becomes

$$(1+i)^{\frac{5}{12}} A_x;$$

other formulas being similarly adjusted.

Assurance Commutation Symbols. It will be observed that as in the case of the annuity, the formulas above given for single premiums would involve much arithmetical work in calculating the values at various ages; but by adopting the same device as that used in connection with annuities, and forming assurance commutation columns, much of the labor is avoided. If the numerator and denominator of the expression for the single premium be multiplied by v^x, we have

$$A_x = \frac{v^{x+1} d_x + v^{x+2} d_{x+1} + \dots\dots\dots}{v^x l_x}$$

and if $v^{x+1} d_x$ be represented by C_x, and $v^{x+2} d_{x+1}$ by C_{x+1}, and so forth, (C being the letter next to D), then we have

$$A_x = \frac{C_x + C_{x+1} + C_{x+2} + \dots\dots\dots}{D_x}$$

And if the series of values of C be summed and represented by M, so that

$$M_x = C_x + C_{x+1} + C_{x+2} + \text{\&c.} \qquad \dots (97)$$

then we have for the single premium the simple expression

$$A_x = \frac{M_x}{D_x} \qquad \dots\dots\dots\dots\dots\dots\dots\dots (98)$$

It is probably unnecessary to go through the form of deducing expressions in Commutation symbols for other assurance benefits. The results may be stated as follows:—

$$A^{1}_{x\,\overline{n}|} = \frac{M_x - M_{x+n}}{D_x} \quad \dots\dots\dots\dots\dots\dots (99)$$

$$_{n|}A_x = \frac{M_{x+n}}{D_x} \quad \dots\dots\dots\dots\dots\dots (100)$$

$$A_{x\,\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{D_x} \quad \dots\dots\dots (101)$$

By using the annuity and the assurance commutation symbols in conjunction we obtain convenient expressions for annual premiums, namely:—

$$P_x = \frac{A_x}{1 + a_x} = \frac{M_x}{D_x} \times \frac{D_x}{N_{x-1}}$$

$$= \frac{M_x}{N_{x-1}} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots (102)$$

$$_nP_x = \frac{A_x}{1 + a_{x\,\overline{n-1}|}} = \frac{M_x}{D_x} \times \frac{D_x}{N_{x-1} - N_{x+n-1}}$$

$$= \frac{M_x}{N_{x-1} - N_{x+n-1}} \quad \dots\dots\dots\dots\dots\dots\dots (103)$$

$$P^{1}_{x\,\overline{n}|} = \frac{M_x - M_{x+n}}{N_{x-1} - N_{x+n-1}} \quad \dots\dots\dots\dots\dots\dots\dots\dots (104)$$

$$P_{x\,\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{N_{x-1} - N_{x+n-1}} \quad \dots\dots\dots\dots\dots\dots (105)$$

It is for such calculations that the inconvenience in the form of the N column is made apparent. In the American notation the expressions are simpler:—

$$P_x = \frac{\mathsf{M}_x}{\mathsf{N}_x}$$

$$_nP_x = \frac{\mathsf{M}_x}{\mathsf{N}_x - \mathsf{N}_{x+n}}$$

&c.

CHAPTER IX.

EXPENSES, LOADINGS AND EXTRA PREMIUMS.

The previous chapter dealt entirely with net premiums and their calculation. But life assurance companies incur considerable expense in maintaining their organization, in writing new assurances, and in collecting the premiums falling due from time to time. They also make returns to the policy-holders of surplus; and such returns have become a recognized part of the system. To the net premiums, therefore, there is added what is called "loading" to provide for such expenses, for surplus, and for other contingencies such as unexpected losses or adverse fluctuations. It is necessary that the premiums charged be sufficient: it is better to have them too large than too small; because, when they are greater than is absolutely essential, an adjustment is made in the surplus distribution, whereas, if they are too small, irretrievable loss may result.

Examples of Net Premiums.

In order to understand better the problem before us in the loading of premiums, it is desirable to have a general idea of the relative magnitude of net premiums for different ages and classes of policies. The following schedule gives the net premiums by the American Experience Table, with 3½% interest:—

AMERICAN EXPERIENCE TABLE 3½%.

KIND OF POLICY.	Specimen Net Premiums per 1,000 Age 20.	Age 40	Age 60
Term Assurance, 5 yrs.............	7.64	9.91	30.15
" " 20 "	8.09	13.23	49.66
Ordinary Life, Continued Payments.	13.48	23.50	56.83
" " Thirty "	16.53	25.42	56.87
" " Twenty "	20.72	30.75	59.85
" " Ten "	34.23	49.78	83.59
Endowment Assurance, 30 yrs......	24.18	28.18	56.91
" " 20 "	38.90	41.18	61.65
" " 10 "	86.30	87.58	99.51
Joint Life Policy, Continued Payments, (two lives both same age)..	21.61	35.45	87.50
Survivorship Assurance, life assured aged 20 in each case—counter life as given at top of column........	10.80	8.90	7.80

Usual Expenses Incurred. It will be seen that the basis formed by the net premium in each case is very variable. The outlays, apart from claims and surrender values, incurred by a life assurance company have to be met by an addition or loading to these net premiums. The usual expenses consist of:—

(1) Salaries of officials and rents at Home Office and branches;

(2) Commissions to agents on first premiums and on renewal premiums;

(3) Medical expenses for the examination of new risks, for the reinstatement of lapsed policies, and for advice relative to settlement of death claims;

(4) Printing and clerical expense, for new policies, for notifying policy-holders that their premiums are due, and for collecting premiums;

(5) Taxes imposed by State Legislatures; and

(6) Expenses in settlement of death claims, and other incidental outlays.

Investment expenses are usually taken care of by the interest income, a rate of interest being assumed in premium calculations

which can be earned net, after meeting such outlays. The commissions payable and the taxes are usually represented by a percentage of the premiums. These expenses, therefore, vary in amount according to the age of the applicant and the kind of policy which may be taken. The outlays for writing policies, medical examinations, mailing notices, settling claims, and in general the Home Office expenses, depend more largely upon the number of policies issued or outstanding; such expense can therefore be more accurately computed on the basis of each $1,000 of sums assured.

Constant Addition Loading.

If a constant addition were made to the net premium, such as 5 for each 1,000 assured, the loading on a Five Year Term policy would be heavy at the younger ages in proportion to the premium, while the loading on a Ten Year Endowment policy would be only a very small proportion of the premium. In the first case, there would be a provision for expenses at young ages of about 40% of the full premium, whereas in the second the expense provision would be less than 6%. It will be seen from the above that part of the expense is quite independent of the amount of the premium. But the commissions are most frequently paid as a percentage of the premium; and taxes are assessed in like manner, while there are other charges which depend almost entirely upon the amount of the premiums collected. The constant addition to the premium does not make adequate allowance for these, and accordingly it is now seldom recommended.

Percentage Addition Loading.

On the other hand, if a percentage addition to the net premium were adopted, such as 30% of the premium, then the cash loading on a Five Year Term policy at age 20 would be only 2.29; at age 60 for the same class of policy it would be 9.05; on an Ordinary Life policy by Continued Payments at age 20 it would be 4.04, whereas at age 60 the loading would be 17.05; and, lastly, on a Ten Year Endowment at age 60 it would be 29.85. Such load-

ings, it will be seen, would vary from 2.29 to 29.85 for each 1,000 assured, according to the kind of policy and age of the life assured. Such policy-holders, therefore, as might enter at the older ages by this system of loading would be paying a much greater cash addition over the net rates than those entering at the younger ages. Moreover, a percentage loading like 30%, added to a Ten Year Endowment policy, would provide much more than is required, and it would make the premium rate most unattractive to any applicant. The system of percentage loading is just as objectionable as that of constant cash additions; because the actual expenses are not confined solely to the one class or to the other, but include both. Some modification or compromise is necessary as between the two.

Customary Method. Notwithstanding the above objections, it is a common custom in the United States to add a fixed percentage to Continued Payment premiums for all ages. A partial adjustment of the equities as between young and old entrants takes place in the method of surplus distribution. If the percentage be 33.3%, then in forming Limited Payment Life and Endowment Assurance rates the common rule is to add 16.7% of the net rate for the particular kind of policy and 16.7% of the Continued Payment net premium. For example, taking the net premium rates at age 40 as given above, the loaded rates would be obtained as follows:—

Continued Payments, Net Rate..........	23.50	
Add 33.3% (say 1/3)..................	7.83	
Office Rate, Continued Payments......		31.33
20 Payment Life, Net Rate.............	30.75	
Add 1/6	5.13	
Add 1/6 of Whole Life Rate...........	3.91	
Office Rate, 20 Payments.............		39.79
20 Year Endowment Net Rate..........	41.18	
Add 1/6	6.86	
Add 1/6 of Whole Life Rate...........	3.91	
Office Rate, 20 Year Endowment......		51.95

The effect of the above method is somewhat similar to that of adding a percentage of the premium and a constant for each 1,000 assured: it makes an adjustment at least as between different classes of policies. But this method does not adjust equitably the premium rates as between young and old entrants.

Constant and Percentage. The two methods of (1) constant addition, and (2) percentage addition, are each incomplete in themselves, but they are to a great extent complementary, so that a combination of the two systems yields good average results and still retains the advantage of simplicity. The percentage provides for an increasing cash loading according to the magnitude of the net premium used as the basis, whereas the constant secures an adequate allowance for administration expenses, which depend to a greater extent upon the sums assured written and remaining in force. For example, premium rates which on the average would prove satisfactory might be computed by adding a constant of 4 for each 1,000 and thereafter 15%, as follows:—

Continued Payment Net Rate, age 40...	23.50	
Add Constant of....................	4.00	
	27.50	
Add 15%	2.75 1.38	
Office Premium by Continued Payments		31.63

The following table gives the premium rates which would be obtained by adopting this uniform rule throughout the same classes of policies for which net premiums were given in the preceding table:—

AMERICAN EXPERIENCE 3½% PREMIUM,

With Loadings: Constant 4 per 1,000, plus 15%.

KIND OF POLICY.	SPECIMEN PREMIUMS PER 1,000, LOADED AGE 20	AGE 40	AGE 60
Term Assurance, 5 yrs.............	13.39	16.00	39.27
" " 20 "	13.90	19.81	61.71
Ordinary Life, Continued Payments.	20.10	31.63	69.95

KIND OF POLICY.	SPECIMEN PREMIUMS PER 1,000, LOADED AGE 20	AGE 40	AGE 60
Ordinary Life, Thirty Payments....	23.61	33.83	70.00
" " Twenty " ..	28.43	39.96	73.43
" " Ten " ..	43.96	61.85	100.73
Endowment Assurance, 30 yrs......	32.41	37.01	70.05
" " 20 "	49.34	51.96	75.50
" " 10 "	103.85	105.32	119.03
Joint Life Policy, Continued Payments, (two lives both same age)..	29.45	45.36	105.23
Survivorship Assurance, life assured aged 20 in each case—counter life as given at top of column........	17.02	14.84	13.57

Rule Not Always Applicable. It is not right to apply rigidly any rule like the above in all circumstances and to all classes, because it is frequently necessary to give consideration to special questions of mortality or expense which may affect individual forms of policies. In order to illustrate this, reference may be made to the class of survivorship assurance. Frequently the older life is a female, and as the mortality amongst females at the older ages is much lighter than that amongst males, it follows that the net premium rates for a survivorship policy when sex is taken into consideration are entirely different from those deduced from a general table for male lives only, like the American Experience. The finer distinctions of this nature relate more strictly to the province of the actuary, and need not be discussed at length here.

The percentage addition is made for the purpose of meeting charges which are greater or less according to the magnitude of the premium. For the collection of premiums it is usual to pay to agents a percentage of the amount actually collected, often 5%. If this and other similar expenses require 10% in all, then in making provision for these it is not sufficient to add merely 10% to the net rate; but the question is to add such an amount that when 10% is deducted the result will be the net figure. *For example*, if we begin with a net premium of 100, and if 10% of this be added, the result will be 110. Then if expenses of 10% are incurred the net premium actually received will only be 99 (that is,

110 — 11). We should therefore treat the question as one of proportion, and state it thus:—

$$90 : 100 :: 100 : x$$

Whence $x = \frac{10000}{90} = 111.11$, from which if 10% be deducted the net premium will remain at 100.

We usually speak of a percenage, but of course it is the same thing if we consider a fixed ratio addition per unit. Thus 10% would be .1 per unit, 5% would be .05 per unit, etc. Therefore in making provision for expenses of the nature of a percentage of the premium, it is necessary to divide by .9 when the expenses are 10%, or by .95 if the expenses are 5%.

To express algebraically the loaded premiums by means of a constant and percentage:—If P represent the net premium, k the constant, and r the ratio addition per unit, the formula for the full premium is

$$P' = \frac{1}{1-r}(P + k) \qquad \text{.................}(106)$$

Initial Expense. One of the principal items of expense is that connected with procuring new applications and writing new business. Those who need life assurance do not flock to the companies and clamor for protection against early death. On the contrary, agents of the companies have to educate the people and prove to them that life assurance is desirable. It is generally considered that the expense of writing new assurance will equal the first premiums from the new applicants. Subsequent premiums, usually called "renewals", are subject to charges for collection, but the general expense applicable to renewal premiums is much less than that chargeable against the new premiums. Account should be taken of this fact in scientific loading; and this may be done by calculating the average expense for new business, K, and forming the loaded premiums thus:—

$$\text{Continued Payments, } P'_x = \frac{1}{1-r}\left(\frac{A_x + K}{a_x} + k\right) \quad \cdots\cdot (107)$$

$$\text{Limited Payments,}\quad {}_nP'_x = \frac{1}{1-r}\left(\frac{A_x + K}{a_{x\overline{n}|}} + k\right) \quad \cdots\cdots(108)$$

with the proper modifications for other classes of policies.

Sprague's Formula. A formula which for many years has been quoted as a standard for non-participating business was published by Dr. Sprague in 1880, viz.:—

$$P'_x = 1.075\left(P_{[x]} + \frac{.01}{a_{[x]}} + .00125\right) \quad \cdots\cdots(109)$$

This is on the basis of each unit assured; on the basis of each 1,000 of assurance the formula would become:—

$$1.075\left(1000\, P_{[x]} + \frac{10}{a_{[x]}} + 1.25\right)$$

which shows that there is a provision for initial expense of 10 spread over the duration of the contract, a constant of 1.25 per 1,000, and to all of this a percentage—7½%—for renewal and other expenses. The formula has been very generally approved.

Select or Aggregate Tables. In Great Britain premium calculations are generally made from select tables, the basis being the net select premium for the risk. The effect of this method is to give to the life assured the benefit of the light mortality rates during the early years of the assurance, whether the selection in any particular company is equal to the average or not. Where the selection is below the average, the net premiums calculated in this way become inadequate. In America, on the other hand, the table most frequently used for all ordinary classes of policies is the American Experience, the standard table prescribed by statute for valuation purposes, and one which does not take into consideration the effect upon the mortality caused by the length of time for which persons have been assured. The table is found to represent with considerable accuracy the rates of mortality experienced in good average companies after the early selection period has passed. The use of a table of this kind,

while it may provide net premiums larger than those strictly necessary, is to give to each company the profit made from careful selection of the lives. There is one class of policy to which an aggregate table is scarcely suited, *i. e.*:—term assurance for a short period where the right to renew is withheld. For such a contract select tables should be employed. In the case of a policy for temporary purposes, granting protection for one year only, the net risk incurred at an average age is probably no more than 60% of the net rate for one year term assurance by the American Table, even after taking into consideration the fact that the mortality on policies of this class is unusually heavy.

Extra Premiums.

Health Under Average. Sometimes it is found that an applicant for life assurance is not quite up to the standard of health required to place him on the books as a good average life, and yet it may appear as a hardship that he should be unable to obtain the assurance protection he desires. For example, a young man whose father or mother may have died of consumption, and whose personal physique is a good average without being unusually robust, is not such a good risk on the average as one whose parents both lived to a good old age. Nevertheless in many cases such persons prove entirely healthy, and a few years later might possibly obtain the assurance protection required without any difficulty.

Occupation Hazardous. Or again, there are certain occupations, such as that connected with railroad service, or military or naval service, in which the average mortality is higher than that of the non-hazardous occupations. In cases of this kind it is frequently desirable to issue a policy at a special premium rate. In the case of occupations the extra premiums can generally be calculated with considerable accuracy, and the usual practice is to add a fixed sum for each 1,000 assured, and to treat these extra premiums as being on the average sufficient to meet the extra hazard. In the case of extra risk arising from physical infirmity

or family history, the question is much more difficult, because it depends to a certain extent on the experience and judgment of the medical officer.

Fixed Extra. There are three systems of dealing with the extra hazard, all of which have special peculiarities. The first is that already mentioned for hazardous occupation, viz.:—adding an extra to the annual premium. This does not usually take into consideration the class of policy which is being issued, and the extra required for one kind of policy may be different from that necessary for another. For example the extra required for a 20 year term policy is quite different from that on a 20 year Endowment Assurance, because the risk on the latter kind of policy is a decreasing quantity, reduced each year by the growing policy value, as explained more fully under the heading "Cost of Assurance". Accordingly, the extra premium required for hazardous occupation should be graded according to the class of policy taken, the heaviest rate being charged for a term policy, a somewhat lower rate for ordinary life, and much lower for a short period endowment assurance.

Rating up. The second is what is known as "rating up"; and this is usually employed in the case of impairment of health, or weak family history. An applicant of the age of x may in such cases be treated for assurance purposes as of the age of $(x+n)$. If the impairment were slight a young man of the age of 25 might be accepted for assurance at the rates applicable to age 30, and his policy thereafter treated as if he were five years older than his actual age; the surrender values and other benefits being those for age 30 at entry. Sometimes the surrender values and other benefits are given as if for age 25 at entry; but this does not seem just, nor in accordance with the spirit of the "rating up". This system is not popular in America. In spite of all apparent flaws an applicant thinks that, with the extra care he takes of himself, his chances of old age are as good as those of other people. Optimism of this kind is natural, it is right, and

it helps to bring about longevity. To be treated as ten years older than the real age is repugnant to human nature; accordingly the lien system, next explained, is much more attractive, since it rewards those who prove themselves to be average risks. Moreover, some of such men, although the average may be less healthy, do live to extreme old age. Therefore, while the mortality tables provide for persons living to the age of about 100, a few people rated up ten or fifteen years might nominally appear older than the oldest age in the table.

Lien Against Policy. The third system is to issue a policy for the sum applied for at the ordinary rate of premium for the age of the applicant, but to stipulate that in event of death during the first year a certain fixed sum should be deducted from the face value of the policy; in event of death during the second year a smaller sum would be deducted, and so on, until after *n* years (*n* frequently being taken as the Expectation of Life) the sum assured would be payable in full on death taking place. This third method is capable of many modifications, and may be applied to almost any form of policy; but the deduction in the first year is often large. In the case of an applicant with a serious impairment, which might nevertheless be adjusted by means of an extra premium or by rating up the age for ten or fifteen years, the lien method applied to a whole life policy is frequently inapplicable, because the deduction from the sum assured in the early years may actually exceed the face value. In theory, as mathematical equivalents, this condition is correct; it is easy to find by algebra the equivalent lien. But in practice such a policy would not appeal to the applicant, nor could the excess of the lien over the value of the Sum Assured be collected in event of early death, although such collection would be necessary in order to fulfill the theory.

Moreover, this system is entirely unfair and unsatisfactory from the company's standpoint in the case of certain peculiar impairments which make themselves felt in the later years of life, such for example as a family history of cancer or gout or a tendency towards obesity in the applicant. The system, there-

fore, only applies satisfactorily to whole life policies when the risk of extra mortality is slight, is immediate, and is likely to be a decreasing quantity.

Lien System and Endowments. But this system may be applied with great effect, and so as to meet nearly all circumstances, if imposed upon endowment assurances. Nearly all the foregoing objections disappear when the endowment assurance policy is used. If the extra risk be deferred, a short-period policy can be issued without lien. On the other hand, if the risk be immediate, as in the case of a young man with a family history of consumption, and only an average physique, or a personal history of appendicitis, a long-period endowment assurance running for as much as thirty years or even more can be issued with a lien to cover the immediate extra hazard. Still further, if the extra be a constant quantity, a decreasing lien over the entire duration of the endowment period furnishes practically the equivalent of a fixed extra when the increasing policy value, and therefore the decreasing mortality risk, is kept in mind.

Foreign Residence. Another common cause for charging an extra premium is foreign residence. Such cases are treated differently by different companies. By many the treatment is the same as that for hazardous occupations, but some companies have special premiums for tropical and for sub-tropical countries, with the rates varying according to age and class of policy. Such rates are obtained after investigating the mortality found to prevail in such districts. The correct system is to have a mortality table applicable to the particular district in which the business is being written, and deduce therefrom the premium rates for different classes of policies. But the material necessary for following this course is not available, and accordingly more or less rough approximations have to be adopted.

This system would be properly applied also in the case of occupation extras; but here again the subdivisions would be so numerous and intricate that the method is scarcely practicable.

Moreover, the same general occupation may be affected by entirely different rates of mortality in its different branches. Consider, for example, the occupation of "miner". Under this class we may have a tin miner, a copper miner, an ironstone miner, a coal miner, or other kinds of miners, all of whom are affected by different rates of mortality, the risk of accident and disease in their several occupations being quite distinct. Indeed even the subdivided form of coal miner is not sufficiently definite; because we find that those who work in hard coal and anthracite mines are subject to different mortality rates from those mining soft coal. Analogous remarks apply to many other occupations, and therefore the rough approximate method is the only one which in the meantime is feasible.

Doctrine of Average. Throughout all questions of actuarial science it must be borne in mind that the doctrine of average applies, and that the lives must be placed in broad general classes. Sub-divisions must not be over-magnified and the general principle forgotten, because after all the fundamental purpose of life assurance is *to protect those who do not prove equal to the average*, although their prospects may be favorable at the time the application is made. Accordingly, there has been a broad-minded tendency on the part of many companies to issue contracts as free from petty restrictions as it is possible to make them. If an applicant is of safe occupation in a healthy district, and has no intention at the time of changing his occupation or going to an unhealthy climate, many companies will now issue a contract free from all restrictions as to residence, travel, and occupation. This is done on the general principle that a comparatively small number of persons do take up hazardous occupations or go permanently abroad. Such cases may be included in the broad average and any additional risk afterwards incurred treated as part of the average mortality. It was on this principle that the latest general investigation of mortality amongst assured lives, the New British Experience, was made. It included all lives resident in Great Britain at the time the policies were taken, and if those persons afterwards proceeded to an unhealthy

climate and died there, their deaths were nevertheless included. Premiums, therefore, calculated from such a table already provide for extra risks incurred in this way.

The tendency amongst life assurance companies is to get rid of restrictions and petty technicalities, and issue policies regarding which there will be no question whatever when the time comes for payment of the claim. Although this system may make the contract slightly more expensive for the policy-holder of unusual health and of first class occupation with a good family record, yet there is no doubt whatever that the system is the best in the end both for policy-holders and for Life Assurance Companies.

CHAPTER X.

EXTENSION OF COMMUTATION COLUMNS—RETURN OF PREMIUMS—JOINT LIVES, ETC.

In this work it is intended to deal only with the simpler forms of life assurance; but there are some elaborations which are so common in every-day practice that even an elementary book would be incomplete if they were entirely omitted. In particular such a question as the return of premiums in event of death is important, as this is made a special feature by one or two companies. Policies on two or more lives are also common, while such transactions as reversionary annuities are easily understood, although they pass a little beyond the usual elementary scope. This chapter will therefore take up two or three of these miscellaneous subjects in order to show at least how the questions should be approached.

Return Premiums. To deal in the first place with the return of premiums in event of death within the first n years of the assurance, it will be readily seen that this is a form of increasing assurance. Sometimes the full premiums are returned in event of death during a limited time, and sometimes a proportion only. If the premium be P, then the sum assured payable at death during the first year will be increased by the one premium paid; in the second year it will be increased by two premiums, that is 2P; in the third year by 3P; and so forth, increasing for each of the n years, after which the face value only will be payable. In order to deal with a problem of this kind in a

simple way it is necessary to investigate an extension of the commutation columns.

The column M_x was formed by the summation of the column C_x, and the same process might be repeated a second time, when the result would be as is shown by the following schedule:—

$$\begin{aligned} M_x &= C_x + C_{x+1} + C_{x+2} + \ldots\ldots\ldots\ldots \\ M_{x+1} &= \quad C_{x+1} + C_{x+2} + \ldots\ldots\ldots\ldots \\ M_{x+2} &= \quad\quad C_{x+2} + \ldots\ldots\ldots\ldots \\ \&c. &= \quad\quad \&c. \end{aligned}$$

Adding these values we get

$$M_x + M_{x+1} + \ldots\ldots = C_x + 2\,C_{x+1} + 3\,C_{x+2} + \ldots\ldots\ldots$$

If we write the left side of the above equation as equal to R_x, we have

$$\begin{aligned} R_x &= M_x + M_{x+1} + M_{x+2} + \ldots\ldots\ldots\ldots\ldots\ldots \\ &= C_x + 2\,C_{x+1} + 3\,C_{x+2} + \ldots\ldots\ldots\ldots\ldots\ldots \\ &= v^{x+1}\,d_x + 2\,v^{x+2}\,d_{x+1} + 3\,v^{x+3}\,d_{x+2} + \ldots\ldots\ldots \end{aligned}$$

Increasing Assurance. This last form shows that R_x would provide 1 at the death of each of the d_x persons; 2 at the death of each of the d_{x+1} persons; and so on, increasing the sum assured by 1 each year. Accordingly, if we divide R_x by D_x we have the single premium for an assurance commencing at 1 and increasing by 1 each year so long as (x) lives:—

$$(IA)_x = \frac{R_x}{D_x}$$

Where (IA) represents an increasing assurance. The corresponding annual premium would be

$$P(IA)_x = \frac{R_x}{N_{x-1}}$$

If we subtract R_{x+n} from R_x we shall have the following result:—

$$R_x - R_{x+n} = M_x + M_{x+1} + \ldots\ldots + M_{x+n} + M_{x+n+1} + \ldots\ldots$$
$$- M_{x+n} - M_{x+n+1} - \ldots\ldots$$
$$= M_x + M_{x+1} + \ldots\ldots + M_{x+n-1}$$
$$= C_x + 2\,C_{x+1} + 3\,C_{x+2} + \ldots + n\,C_{x+n} + n\,C_{x+n+1} + \ldots$$

which shows that the assurance would be continued after age $x + n$ for the amount to which it had then increased, but that the annual increase in the assurance would no longer take place. In order, therefore, that we may have an assurance increasing for n years and *thereafter reduced to the original figure*, as in the case of the return-premium policies, we must, at the end of n years, not only subtract R_{x+n} but also subtract nM_{x+n}, which will cancel all terms from age $x + n$ onwards in the above expression.

If we express the annual premium for a return-premium policy by the symbol $P(VA)_x$, where (VA) stands for "varying assurance", we shall have—

$$\text{Value of the net premiums} = P\,(VA)_x \times \frac{N_{x-1}}{D_x}$$

$$\text{Value of the Benefit} = \frac{M_x}{D_x} + P\,(VA)_x \frac{R_x - R_{x+n} - n\,M_{x+n}}{D_x}$$

But the value of the net premiums must equal the value of the Benefit, and accordingly we have

$$P\,(VA)_x \frac{N_{x-1}}{D_x} = \frac{M_x}{D_x} + P\,(VA)_x \times \frac{R_x - R_{x+n} - n\,M_{x+n}}{D_x}$$

$$\text{whence}\quad P\,(VA)_x = \frac{M_x}{N_{x-1} - (R_x - R_{x+n} - n\,M_{x+n})}$$

Premium Return in Practice. The above method deals with the question as one of theory, in which the *net* premium only is found, and the net premium returned to the policy-holder. Policy-holders generally know nothing about net premiums, and deal only with gross premiums; indeed, no reference whatever is made in the policy contract to any premium other than the gross. Moreover, after the premium-return period expires, it is usual to reduce the annual premium to that for an ordinary policy taken

at the original age. The practical question therefore differs from the theory. It reduces itself to calculating the *additional* payment necessary for providing the premium-return feature, so as to leave the ordinary policy on its original basis. If, therefore, the gross annual premium for a Whole Life policy be represented by P'_x, and if the net additional premium for the return feature be represented by P, and if the loading on this net addition be in the ratio k, so that the total addition to the ordinary premium would be $P(1+k)$, then in calculating P, the *net* additional premium, we have to provide for the return during n years of the total annual premium of $P'_x + P(1+k)$. This may be done as follows:—

The total net additional payments in the n years would be.................... $P\,\dfrac{N_{x-1} - N_{x+n-1}}{D_x}$

The benefit for premium-return would be

$$\left\{P'_x + P(1+k)\right\} \frac{R_x - R_{x+n} - nM_{x+n}}{D_x}$$

By equating the payment and the benefit we may solve for P as follows:—

$$P\,\frac{N_{x-1} - N_{x+n-1}}{D_x} = \left\{P'_x + P(1+k)\right\} \frac{R_x - R_{x+n} - n\,M_{x+n}}{D_x}$$

whence

$$P = \frac{P'_x\,(R_x - R_{x+n} - n\,M_{x+n})}{(N_{x-1} - N_{x+n-1}) - (1+k)\,(R_x - R_{x+n} - n\,M_{x+n})}$$

The above question is taken up in some detail, as it affords a good illustration of the extension of commutation columns and the more difficult problems which may arise in connection with life assurance. Many variations can be made on a problem of this kind, seeing that it may be applied in the case of Limited Payment or Endowment Assurance, and in other circumstances.

Bonus Additions. Increasing assurances are common in Great Britain, where bonuses are frequently allocated in the form of an addition to the sum assured for each year the policy is in force. If the rate of this annual addition for a whole-

life policy be b, then we have as the net premium to provide for the assurance commencing at 1 and increasing by b for each annual premium paid—

$$P = \frac{M_x}{N_{x-1}} + \frac{b\,R_x}{N_{x-1}}$$

The annuity commutation column N_x may be summed, forming a column S_x, bearing the same relationship to N_x which R_x bears to M_x. This column, S_x, would enable us to solve problems involving an annual increase or decrease in the premiums. Contracts of this nature are seldom required, and there is therefore no need to elaborate this part of the subject.

Complete Annuities. The annuity values in Chapter VII. were investigated on the understanding that the last payment would be the regular periodic sum which the annuitant might live to receive: no proportion would be paid from the date of such periodic payment to the exact date of death. But sometimes it is desired that such proportion be paid, and the sum then payable (on the assumption that deaths occur at equal intervals over the year) would average half of the annual payment, and this sum would be due at the moment of death. The present value of this additional payment would therefore be ½ $\overline{A}_x$, that is the single premium to assure ½ payable at the moment of death; and the total value of the complete annuity is

$$\begin{aligned}\mathring{a}_x &= a_x + \tfrac{1}{2}\,\overline{A}_x \\ &= a_x + \tfrac{1}{2}\,A_x\,(1+i)^{\frac{1}{2}} \quad \ldots\ldots\ (110)\end{aligned}$$

If the annuity were payable half-yearly, then in like manner, on the assumption that deaths occur at equal intervals over each half-year, the proportionate payment to the date of death would on the average be ¼. Accordingly, the half-yearly annuity with proportion to date of death would be

$$\mathring{a}_x^{(2)} = a_x^{(2)} + \tfrac{1}{4}\,A_x\,(1+i)^{\frac{1}{2}}$$

The general algebraic expression if the annuity were payable m times a year would be

$$\overset{\circ}{a}{}^{(m)}_{x} = a^{(m)}_{x} + \frac{1}{2\,m}\,\bar{A}_x$$

$$= a_x + \frac{m-1}{2\,m} + \frac{1}{2\,m}\,A_x\,(1+i)^{\frac{1}{2}} \qquad \ldots\ldots.(111)$$

The expression $(1+i)^{\frac{1}{2}}$, which gives the amount of 1 in half a year at compound interest, may be written as approximately equal to $(1+\frac{i}{2})$, which is the amount of 1 in half a year at simple interest: the difference between the two for a short term is very small and may be ignored.

Joint Lives.

Annuities. In Chapter VI. probabilities were investigated relating to two lives in combination, and expressions given for the probability that both (x) and (y) would live n years. By combining with these probabilities the present value of money we can obtain a simple expression for the annuity to continue so long as (x) and (y) survive, namely:—

$$a_{xy} = vp_{xy} + v^2\,{}_2p_{xy} + v^3\,{}_3p_{xy} + \ldots\ldots\ldots.$$

$$= \frac{vl_{x+1\,:\,y+1} + v^2 l_{x+2\,:\,y+2} + \ldots\ldots\ldots\ldots}{l_{xy}}$$

It should be observed particularly that this annuity is payable only while *both* (x) and (y) are alive; it terminates with the first death of the two. This form of annuity is commonly required in connection with joint-life assurances where the sum assured is payable at the first death, but in the purchase of annuities the more frequent form furnishes a payment until the death of the survivor, these annuities being occasionally purchased for the benefit of husband and wife. The value of this can be easily expressed, as it consists of a payment during the lifetime of (x) to continue so long as he lives, a similar payment during the lifetime of (y), less a payment while both are alive, since the annuity during the joint lifetime remains exactly the same as it is during the lifetime of the survivor. The formula for this annuity is therefore written:—

$$a_{xy} = a_x + a_y - a_{xy}$$

The same process may be applied to three or more lives, but the circumstances under which such contracts are desirable are comparatively rare. To facilitate calculation, commutation columns for two or more lives can be formed, and in other respects the investigations are similar in their nature to those for single lives.

Assurances. It is clear that the verbal reasoning explaining formula (74) is true whether there be one life only or two or more joint lives, and accordingly we can obtain the single premium for an assurance payable on the first death, namely:—

$$A_{xy} = 1 - d\,(1+a_{xy})$$

The corresponding annual premium is

$$P_{xy} = \frac{A_{xy}}{1 + a_{xy}}$$

which may be expressed conveniently in terms of annuities thus:

$$P_{xy} = \frac{1}{1 + a_{xy}} - d$$

Reversionary Annuity. The Reversionary Annuity provides an income to (x) after the death of (y), and involves a joint life function. In this case (x) would receive no benefit during (y)'s survivance, and would get nothing whatever if he should die in the lifetime of (y). The value of this annuity is the difference between two annuity values—one on the life of (x), and one on the joint lives of (x) and (y), that is:—

$$a_{y|x} = a_x - a_{xy}$$

The annuity payments to (x) provided by a_x would be entirely cancelled during the joint lifetime of (x) and (y) by the subtraction of a_{xy}. Thereafter they would be left free for (x) after (y)'s death. If (x) were to die first, then the two annuities would simultaneously terminate, as is proper in the circumstances.

Arithmetical examples of such annuities may be an aid to proper understanding. The value of an annuity of 1 on a single life at age 30 (American Experience 3½%) is 18.605. The Joint

Annuity for two persons aged 30 and 60 would be 9.311. Therefore an annuity of 1 to a person of age 30 after the death of another aged 60 would be 9.294. A single life annuity at age 60 would cost 10.032, and therefore a reversionary annuity to (60) after (30) would cost .721—a small sum, because the chance of a person of 60 surviving a person of 30 is small. For the protection of any individual, such for example as a son making provision for his mother or father, this form of transaction when understood is excellent; but it is far from popular, because most persons who effect policies of assurance object to the possibility of losing all that is paid in—forgetting that for this very reason they pay so much less.

CHAPTER XI.

VALUATION OF POLICY LIABILITIES.

As the life assured under a policy grows older the date for payment of the sum assured approaches, because "the young *may* die, but the old *must*". The liability for payment of the sum assured therefore increases. Moreover, with increasing age the value of the premiums still to be received by the company becomes less, unless the premiums increase under a Renewable Term contract. In the extreme limit, when the life assured has attained the oldest age in the mortality table, the sum assured is just about to fall due and the future premiums are worth practically nothing. At that time therefore the company must have the sum assured in hand so as to be in a position to pay the claim. The fund for this purpose is accumulated gradually during the currency of the policy, and hence it is that life assurance companies on the old-line plan must hold large funds with which to meet the obligations under their contracts.

"Reserve" Misleading Word. The liability under these obligations is often spoken of as the "reserve", but this is an unfortunate use of the word. The fund held by the company is an *ascertained liability;* it is not a provision against possible or unforeseen losses. The same word is used in Banking to denote the fund held over and above the deposits as an additional protection to them. If "reserve" were used in an analogous sense in Life Assurance it would mean the same as "surplus". The liabilities of an assurance company are of a nature similar to bank deposits, though in one important respect they differ:

they are held for the general protection of all, and should not be considered as belonging to individuals, like bank deposits. The total liability represented by the assurance fund is correct *on the average*, but in individual cases it may prove either insufficient or excessive. The death of a policy-holder may at once increase the liability in his individual case from a sum like $500 to as much as $10,000, this increase being met out of the general fund. The words "policy value" are therefore used herein to denote the liability under any particular policy, and this has the further advantage of adhering more closely to the notation where the letter V is used, being the initial letter of the word "Value".

Whole-Life Policy Value. We have seen that when a policy is first taken, the mathematical value of the future net premiums is exactly equal to the value of the sum assured

$$P_x\ (1+a_x) = A_x$$

But as the life assured grows older, and as the annual premium payable under an ordinary life policy remains the same, the value of the future premiums becomes less, while with increasing age the value of the sum assured becomes greater as the date when payment must be made approaches. Accordingly, the equation no longer holds, but

$$P_x\ (1+a_{x+n}) < A_{x+n}$$

The difference between the two sides of this expression is the net value of the policy after n years. It is the average sum which the company should have in hand as liability under a policy, so that it may continue to provide the same protection without increasing the premium with advancing age. This net value is expressed as ${}_nV_x$, and we therefore have

$${}_nV_x = A_{x+n} - P_x\ (1+a_{x+n}) \qquad \ldots\ldots(112)$$

This is called the *prospective* method of valuation because it looks entirely to the future. A_{x+n} is the value of the liability for sum assured to be paid, while $P_x\ (1+a_{x+n})$ is the value of the future premiums to be received. The general principle applies to all

forms of policies, that the difference between the increased value of the sum assured and the decreased value of the future premiums represents the policy value. The other method of obtaining policy values is known as the *retrospective* method, and consists in accumulating the net premiums paid under the contract and deducting therefrom the accumulated claims by death. Needless to say the results are exactly the same when net premiums are used.

Mathematical Transformations. The above formula may be transformed mathematically and stated in other ways, the study of which is both interesting and instructive. As the annual premium, the single premium, and the annuity value can each be expressed in terms of the others, we can therefore in like manner give the policy value in terms of any one, or any two, of these functions as well as all three:—

$$\begin{aligned} {}_nV_x &= A_{x+n} - P_x\ (1+a_{x+n}) \\ &= P_{x+n}\ (1+a_{x+n}) - P_x\ (1+a_{x+n}) \\ &= (P_{x+n} - P_x)\ (1+a_{x+n}) \qquad \ldots\ldots\ldots\ (113) \end{aligned}$$

The foregoing formula shows that the policy value is equal to the difference for the remainder of life between the net premium which a person at the attained age $x+n$ would pay, and the net premium which he actually has to pay through having entered at age x.

Again, as $(1+a_{x+n}) = \dfrac{1}{P_{x+n}+d}$

we have $${}_nV_x = \frac{P_{x+n} - P_x}{P_{x+n}+d} \qquad \ldots\ldots\ldots\ (114)$$

giving the value of a policy in terms of annual premiums with the rate of interest. This shows also that the value of the policy is dependent upon the ratio of increase in the premium payable for a Whole Life Policy as between ages x and $x+n$.

Another interesting expression is obtained as follows:—

$$\text{As} \quad P_{x+n}\ (1+a_{x+n}) = A_{x+n}$$

$$\therefore\ 1+a_{x+n} = \frac{A_{x+n}}{P_{x+n}}$$

$$\text{and we have} \quad {}_nV_x = A_{x+n} - P_x \times \frac{A_{x+n}}{P_{x+n}}$$

$$= A_{x+n}\left(1-\frac{P_x}{P_{x+n}}\right) \quad \ldots\ldots(115)$$

The expression inside the brackets represents the portion of the sum assured which has been already purchased by the premiums paid between ages x and $x+n$. If a person of the age $x+n$ were to approach an assurance company and offer to pay a premium of P_x only, the sum assured to which he would be entitled would be $\frac{P_x}{P_{x+n}}$; he would not be entitled to the full sum assured of 1, but only to this proportion of it. Therefore, when a person has the right to continue his policy at age $x+n$ for the premium of P_x, it follows that the sum assured actually purchased by his past payments must be $1-\frac{P_x}{P_{x+n}}$, as above. This is a formula for the paid-up value after n years, and the single premium to provide this paid-up policy is of course $A_{x+n}\left(1-\frac{P_x}{P_{x+n}}\right)$, which is also the net value of the ordinary life policy.

To write the whole expression in terms of annuities, we have—

$$
\begin{aligned}
{}_nV_x &= A_{x+n} - P_x(1+a_{x+n}) \\
&= 1-d\,(1+a_{x+n}) - \left(\frac{1}{1+a_x} - d\right)(1+a_{x+n}) \\
&= 1-d\,(1+a_{x+n}) - \frac{1+a_{x+n}}{1+a_x} + d\,(1+a_{x+n}) \\
&= 1-\frac{1+a_{x+n}}{1+a_x} \\
&= \frac{a_x - a_{x+n}}{1+a_x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(116)
\end{aligned}
$$

This is a useful formula, which is used frequently for computing tables of policy values, and from which isolated policy values may be conveniently calculated. It shows that the decrease in the annuity value caused by increasing age is responsible for the fact that a policy value is necessary.

Other Classes. Exactly the same theories may be applied to find the policy value for other classes of assurance. For the Limited Payment policy, where r represents the number of years during which premiums were to be limited, we have

$$_{n|r}V_x = A_{x+n} - {}_rP_x\,(1 + a_{x+n:\overline{r-n-1}|}) \quad \ldots\ldots(117)$$

As there are only $r - n$ premiums still payable, and one of these is just due, the annuity takes the form as given. When n becomes equal to r, the policy is paid-up for life and we have

$$_{r|r}V_x = A_{x+r}$$

which is evidently as it should be.

For an Endowment Assurance running for r years the value is

$$_nV_{x\overline{r}|} = A_{x+n:\overline{r-n}|} - P_{x\overline{r}|}(1 + a_{x+n:\overline{r-n-1}|})$$

Terminal and Mean Values. All the foregoing values are what are known as "Terminal Values", *i. e.*:—the value of the policy at the date when a premium is due, and just prior to payment of the premium. If the value be calculated at the same date but *after* the premium has been paid, it is called the "Initial" value, and is equal to the terminal value plus the net premuim just paid; *i. e.*:—

$$\left.\begin{array}{r}\text{Initial Value}\\ \textit{after } n \text{ years}\end{array}\right\} = {}_nV_x + P_x$$

In the valuation of a Life Assurance Company, policies are valued at one fixed date, usually 31st December. As they have been issued at intervals throughout the year, some of them having the policy year running from January, others from February, and so on, terminal values are not applicable on 31st December. It might be possible to compute the value of each policy to the nearest month, or even with extreme accuracy to the exact day, by proportioning between the initial and terminal values. But this would be a most tedious operation, and as policies are issued at all times during the year, the assumption is made in America that on the average they are issued about the

middle of the year. The common practice, therefore, is to use on 31st December what are known as "Mean Values". The Mean Value is simply the average between the initial and terminal values as above explained. For an ordinary Whole Life policy, therefore, we can express

$$\text{Mean Value for } n^{th} \text{ year} = \frac{{}_{n-1}V_x + P_x + {}_nV_x}{2} \qquad (118)$$

and the

$$\text{Mean Value for } n + 1^{th} \text{ year} = \frac{{}_nV_x + P_x + {}_{n+1}V_x}{2}$$

For valuation purposes tables of Mean Values have been computed, and they are almost invariably employed. To compute such mean values it is usual first to calculate the terminal values, so that the study of terminal values is much the more important; and, moreover, terminal values are properly adopted for surrender and loan calculations and for other general purposes where individual policies are concerned, because such calculations are made as at the date of renewal.

Net Valuations. The equation connecting the value of the sum assured with the value of the premiums, viz.:—

$$A_x = P_x\,(1+a_x)$$

assumes that all three functions, A_x, P_x, and a_x, are based upon the same mortality table, and taken at the same rate of interest; and, when this is so, the equation holds rigidly. But the premiums actually payable under any policy are not the net premiums, but loaded or gross premiums. If the "loading" over the net premium be represented by k, then the value of the future gross premiums would be $(P_x+k)\,(1+a_x)$, and this is no longer equal to the value of the sum assured at the date the policy is taken, but is in excess. The excess value $k\,(1+a_x)$ represents the provision for future expenses and surplus, apart from savings which may accrue from interest earnings exceeding the rate assumed, from favorable mortality experience, etc.

Sometimes the net premiums which form the basis for the gross premiums have been taken from a different mortality table

and at a different rate of interest from that used in valuation. Nevertheless the custom in ascertaining policy liabilities is to take the single premium, the annuity, and the net premium, all from the same mortality table and at the same rate of interest. In the insurance laws it is usual to neglect the question of loading, on the ground that companies are at liberty to charge what premiums they consider right. The only restriction is that the premium must be sufficient, in the case of old-line assurance, to provide the net benefit.

Contract Premium Ignored. The net premium method of valuation was first introduced to correct abuses which had crept into the practice of policy valuations, because several companies under the laws in force forty or fifty years ago made practically no provision whatever for future expenses, but took credit for the entire gross premiums to be received in future. The net premium method does not take into consideration *in any way* the gross premium under the contract; and, in terms of the valuation laws in America, no account is taken of the contract premium so long as such premium is equal to, or greater than, the net premium. The method, therefore, does not take into consideration one of the principal factors connected with the actual contract in force. The sum assured is a guaranteed amount, and the gross premium is a fixed quantity for any specific contract. On the other hand the net premium is theoretic, depending upon the mortality table and rate of interest employed. Of course some provision, and a full and adequate provision, must be made for future expenses in the valuation of a life assurance company. One certain way of doing this is to employ the actual contract premiums, reduced by such allowance for future expenses as may be amply sufficient; but it is open to this serious objection, that occasionally the value of the premiums may be greater than that of the obligation for payment of the sum assured, and in that case the policy would have a *negative value*. In other words, the policy would appear as an asset of the company instead of a liability, and such assets cannot be considered good because the policies might lapse at

any time, and these assets would thereby be lost. If the contract premiums closely correspond with the net premiums, then, by employing the latter for ascertaining policy liabilities, an insufficient provision for the future may be made. On the other hand, if there is a large margin between the contract and the net premiums, the use of the latter in valuation might bring out *apparent* weakness in the case of a company actually strong, healthy, and vigorous. Whatever may be the best system for arriving at the true liabilities, the fact must not be forgotten that American Statutes at the present time provide for the use of net premiums.

Although the contract premium is left out of account in valuation, various devices have been used for making allowance for the great difference in expense between the first and renewal years. It was shown in Chapter VII. that initial expenses are necessarily high, also that this fact is frequently taken into consideration in calculating premiums. The method of valuation should properly and logically follow the method of premium calculation; the gross premium, the net premium, and the loading, should all form part of the calculation, due consideration being given to the reasons for making additions to the net premiums, whether for initial expense, for renewal expense, or for dividends or bonuses. But this would have the effect of making the valuations unduly intricate, and simpler methods have accordingly been devised.

Preliminary Term Valuation. One of these methods is to assume that all the first year's premium is used in initial expenses and risk, treating that year as furnishing term assurance only. The policy is then renewed one year later under the form chosen by the applicant, and the regular policy values commence with the second year. For example, if a policy were taken at age 35 by annual premiums for life, the first year's premium would be for term assurance, the net premium for which by the American Experience Table at 3½% is 8.63 per thousand, and the Whole Life Policy would go into effect at age 36, at which age the net premium for a Whole Life Policy is 20.55. Accordingly, if the contract premium were $27.00, the loading in the

first year would be $18.37, and in subsequent years $6.45. This method therefore has the effect of providing a larger loading the first year, and a smaller loading afterwards. The policy contracts are usually written in such a way as to show that this is the intention of the parties.

When the same principle is applied to Limited Payment Policies and Endowment Assurances, there are serious objections to this plan, because the allowance for first year's expenses then becomes extravagant. It involves the assumption in the case of a short period Endowment Policy that a premium of as much as 100 for each 1,000 assured would be paid for one year's temporary assurance; or that the loading for expenses in that premium is $91.37 as compared with $18.37 in the above example of a Whole Life Policy. The allowance for the initial expense is therefore much too great. In writing an Endowment Policy the commissions and charges should exceed by only a very small sum the expense of writing a Whole Life Policy. If therefore $18.37 per thousand is a reasonable expense for writing Whole Life Policies, the same, or but little more, should be allowed for writing Endowment Assurances.

Modified Preliminary Term.

To overcome the objections to the preliminary term method above outlined, it has been suggested that, as the preliminary term plan is satisfactory and reasonable in the case of Whole Life Policies, the same allowance should be made for other classes. If therefore $18.37 per thousand were considered the proper expense allowance in the one case, the same amount would be allowed for Limited Payments and Endowment Assurances. Under this plan the Whole Life premium at any age forms the measure of the expense which can be incurred at that age for Limited Payment and Endowment Assurances; the expense allowance is therefore an increasing amount according to the age of the applicant. This principle has been recognized, although not to its full extent, by the recommendations of the "Committee of Fifteen", a Committee of Governors and Insurance Commissioners, who desire to introduce better insurance laws, and to bring the laws of the

different States in the United States into harmony. They recommend that the preliminary term method must be "modified" for all Endowment Assurances and for all Limited Payments *where the premiums are less than twenty.* For Twenty Payment Policies and those having a longer premium-paying period, the full preliminary term method has been approved.

Select and Ultimate Valuation. Still another suggestion has been made for meeting the same conditions, and this suggestion is of great importance as it has been adopted by the Legislature of the State of New York. The scientific principle involved is that the benefit of introducing new amongst old policy-holders lies in the light mortality of the new for several years, as well as in their sharing of expenses. The latter advantage is an incident in the expansion of all businesses; the former is peculiar to life assurance, and that benefit may reasonably form the measure of expense which a company is justified in incurring. Moreover the expenditure will be made good to the company within a period of five years. The special method of valuation therefore consists in allowing a company to take credit for the probable savings in mortality due to effective medical selection. The provisions of the New York Statute are:—

> "The legal minimum valuation of all contracts issued "on or after the first day of January, nineteen hundred "and seven, shall be in accordance with the select and "ultimate method, and on the basis that the rate of mor-"tality during the first five years after the issuance of said "contracts respectively shall be calculated according to "the following percentages of the rates shown by the "American experience table of mortality, to wit,
> "1st insurance year 50 per centum thereof,
> "2d insurance year 65 per centum thereof,
> "3d insurance year 75 per centum thereof,
> "4th insurance year 85 per centum thereof, and
> "5th insurance year 95 per centum thereof."

The difference between the mortality assumed by the use of the American Experience Table, and the actual select mortality, as represented by the percentages above mentioned, during the

first five years, is discounted and credited to the company as an offset against the reserves. Premiums are computed by the use of an ultimate mortality table, and the liability for the sum assured and the relative asset (namely, the value of the future premiums) are computed by the use of select functions. The formula for valuation is therefore during the first five years:—

$$_{n}V_{x} = A_{[x]+n} - P_{x}\,\mathbf{a}_{[x]+n}$$

This method gives very satisfactory results and at the same time makes a reasonable allowance for first year's expenses. After five years the policy values are the same as if the device had not been employed.

Cost of Assurance, or Death Strain.

When a death occurs after a policy has been for some time in force, the sum assured forms a disbursement by the company; but, on the other hand, the liability for the policy value disappears, the policy being taken off the books. A claim by death, therefore, has a varying effect upon the resources of the company according to the magnitude of the policy liability which is cancelled, relative to the face value paid. In the case of an Endowment Assurance policy nearing maturity, death causes a very small proportionate strain on the resources of the company, because it only means the payment of the sum assured a short time earlier than would otherwise be the case. The actual strain on the funds of the company should be measured, therefore, not by the sums assured, but by the difference between the sums assured and their policy values. It may happen that the claims by death appear severe, but, on account of their occurring chiefly amongst old policy-holders, the strain on the company may be comparatively light. The true sum at risk is represented by the difference between the face value payable at death and the policy value held by the company, and therefore the "Cost of Assurance" at any age should be measured by this difference. If S represent the sum assured, and V the policy value, we have as the cost of assurance the general formula q_x (S—V). This function, called the "Cost of Assurance" in America, is usually referred to in Great Britain as the "Death Strain".

Numerical Illustrations. A numerical example will make the above much more clear. If a policy be taken out on the Whole Life Plan at age 35, the company must in ten years accumulate a reserve of $136 (American Experience 3½%). This amount is held as a liability under the policy contract. If death were to occur in that year the sum assured of $1,000 would be paid, but the liability for the policy value of $136 would be cancelled. The net amount taken from the general mortality fund of the company would therefore be only $864. If the same policy ran for ten more years the policy value would be increased to $311, and the net payment at death after that time would only be $689. After thirty years the policy value would be $505 and the net mortality payment $495, and so forth.

Therefore the true cost of assurance, or mortality cost, is measured, as above indicated, not on the face value of the sum assured but only on the net amount withdrawn from the mortality fund in event of death. This net amount varies greatly according to the kind of policy under consideration. Under a Twenty Year Endowment Assurance taken at age 35 the policy value at the end of ten years would be $396. The net payment at death in that year would therefore be only $604, as against $864 under a Whole Life Policy. At the end of the fifteenth year the Endowment policy value would be $664 and the net mortality payment $336. At the end of the twentieth year the policy value is $1,000, so that in event of death during that year the company would have no strain whatever on the mortality fund on account of such death.

CHAPTER XII.

SETTLEMENTS AND SURPLUS.

The settlement contemplated when a policy is effected is that which is made when a claim by death occurs, or when an Endowment Assurance matures. But a great many policies terminate in other ways, the largest number of all ceasing at the end of the first year of the policy through lapse. The expenses incurred in obtaining the application, conducting the medical examination, and issuing the policy are heavy, and companies do not pay any surrender values if policies lapse at the end of the first year. After the first year, however, expenses are smaller, and it is generally understood that the light mortality arising through the selection of healthy lives will enable a company to make good its initial expense, and set aside the regular policy value from the premiums received, in a period of not more than five years from the date of issue of a policy. Accordingly, in event of lapse, sometimes after the second year but more frequently after the third, an equitable surrender value is allowed in cash or in paid-up assurance or extended assurance.

Legal Surrender Values. Insurance laws in the United States provide that the policy-holder must be allowed an equitable proportion of the policy value of his own contract, and various enactments have been passed to protect his interests. In most instances the true situation has been recognized, that the full policy value does not belong to individual policy-holders, as was mentioned in the last chapter, and the laws generally allow

some compensation to the companies when policy-holders decide to withdraw. The law of the State of New York formerly read:

> "Whenever any policy of life insurance * * * after "being in force three full years, shall, by its terms, lapse "or become forfeited for the non-payment of any premium, "* * * the reserve on such policy computed according "to the American experience table of mortality at the rate "of four and one-half per cent per annum shall, on demand "made, with surrender of the policy within six months "after such lapse or forfeiture, be taken as a single pre-"mium of life insurance at the published rates of the cor-"poration at the time the policy was issued, and shall be "applied, as shall have been agreed in the application or "policy, either to continue the insurance of the policy in "force at its full amount so long as such single premium "will purchase temporary insurance for that amount, at "the age of the insured at the time of lapse or forfeiture, "or to purchase upon the same life at the same age paid-up "insurance payable at the same time and under the same "conditions, except as to payments of premiums, as the "original policy".

The law further provided that the policy contract could specify which one of these two methods must be chosen, and instructed that the net surrender value should in no case be less than two-thirds of the entire policy value computed by the above rules.

The new law of New York is entirely different with regard to surrender values, and it provides that although no claim for value be made the company must nevertheless continue the policy as extended term assurance for its face amount less indebtedness, if any. The law now reads thus:

> "If any policy of life insurance (other than a term policy "for twenty years or less), issued on or after January first, "nineteen hundred and seven, by any domestic life insur-"ance corporation, after being in force three full years "shall by its terms lapse or become forfeited by the non-"payment of any premium or any note therefor or any "loan on such policy or of any interest on such note or "loan, the reserve on such policy computed according to "the standard adopted by said company in accordance "with section eighty-four of this chapter, together with "the value of any dividend additions upon said policy, af-

"ter deducting any indebtedness to the company and one-"fifth of the said entire reserve, or the sum of twenty-five "dollars for each one thousand dollars of the face of said "policy if said sum shall be more than the said one-fifth, "shall upon demand with surrender of the policy be ap-"plied as a surrender value as agreed upon in the policy, "provided that if no other option expressed in the policy "be availed of by the owner thereof, the same shall be "applied to continue the insurance in force at its full "amount including any outstanding dividend additions "less any outstanding indebtedness on the policy, so long "as such surrender value will purchase nonparticipating "temporary insurance at net single premium rates by the "standard adopted by the company, at the age of the in-"sured at the time of lapse or forfeiture, provided in case "of any endowment policy if the sum applicable to the "purchase of temporary insurance shall be more than "sufficient to continue the insurance to the end of the "endowment term named in the policy, the excess shall "be used to purchase in the same manner pure endow-"ment insurance payable at the end of the endowment "term named in the policy on the conditions on which the "original policy was issued, and provided further that any "attempted waiver of the provisions of this paragraph in "any application, policy or otherwise, shall be void, and "provided further that any value allowed in lieu thereof "shall be at least equal to the net value of the temporary "insurance or of the temporary and pure endowment in-"surance herein provided for. The term of temporary in-"surance herein provided for shall include the period of "grace, if any."

Massachusetts Surrender Values.

In the State of Massachusetts the law governing domestic companies (*i. e.*, companies incorporated in the State of Massachusetts) provides that after three full annual premiums have been paid under a life or endowment assurance,

"in case of default in the payment of any subsequent pre-"mium, then without any further stipulation or act such "policy shall be binding upon the company for the amount "of paid-up insurance which the then net value of the "policy and all dividend additions thereon, * * * less "any indebtedness * * *, and less the surrender

"charge provided herein, will purchase as a net single pre-"mium for life or endowment insurance maturing or "terminating at the time and in the manner provided in "the original policy contract * * * . Said surrender "charge, unless fixed at a smaller rate by the policy, shall "be five per cent of the present value of the future net "premiums at the date of default, which by its terms said "policy is exposed to pay in case of its continuance. * * * "Every such paid-up policy shall have a cash surrender "value, which shall be its net value, less any indebtedness "to the company on account of said policy, and every "policy which by its own terms has become paid up shall "have a cash surrender value, which shall be its net value, "less five per cent of one net premium, and the holder of "any paid-up policy may upon any anniversary of its issue "surrender the same and claim and recover from the com-"pany such surrender value in cash".

The laws of the various states differ materially in the matter of the surrender values which must be paid to policy-holders. Most of them provide only for paid-up or extended assurance, but occasionally, as in the above instance of Massachusetts, a cash value is also made obligatory. Decisions in the Courts of Kentucky have been given to the effect that any value under a policy may be claimed at any time within five years from the date of lapse, notwithstanding any stipulation in the policy contract to the contrary.

Liberal Practices. But the policies of most companies grant to policy-holders even greater privileges than those provided by the state laws. Under the policy contracts of some companies paid-up or extended assurance of greater value than the legal minimum is granted automatically and without request on the part of the policy-holder, so that the policy remains in force either for its full amount for a temporary period, or for a reduced amount as Paid-up Life, or Paid-up Endowment Assurance.

Non-Forfeiture. The general meaning of the expression "non-forfeiture" is that the value of a policy is not forfeited or lost on default in payment of any premium. There are

several different ways of accomplishing this result, all of which are spoken of as non-forfeiture provisions. One form provides that the sum assured is at once reduced in amount and paid-up assurance granted. This paid-up assurance is usually on the same form as the original policy, except that no further premiums are payable; and, in arriving at the amount, the policy value available is used as a single premium to purchase such amount of reduced assurance as the value will provide. This is the form of the Massachusetts law. Another form of non-forfeiture provision continues the policy for its full face value as "extended assurance". In that case, the available policy value is applied to purchase temporary assurance protection for the face value of the policy for such time as the risk can be carried from the attained age of the life assured, as is done under the New York law.

One or two companies have also incorporated a non-forfeiture provision, by which any unpaid premium is advanced as a loan against the policy, and this may be repeated from year to year so long as the loan value under the contract is in excess of the accumulated amount of the indebtedness thus incurred. This provision is also automatic, and the entire rights of the policy contract are preserved for the assured.

Loans. Most policies provide that the company shall grant loans to an amount agreed upon, and within the policy value. By this means policy-holders may tide over temporary difficulties; and the convenience is one which is very generally appreciated by the public. Life assurance companies desire to treat their policy-holders in the most liberal manner possible. This is in the interest of the company as well as that of the policy-holders, because life assurance depends so largely for its success on the good will and friendship of the public. The loan privileges are much more convenient than the usual conditions of borrowing. The loan may be obtained on the shortest notice, time being required only for the preparation and signature of the necessary papers, and occasionally this can be done within an hour. No expense is incurred by the borrower for legal advice or any

other service, a very moderate interest rate is charged, and the advance may generally be ***repaid at any time without notice*** on payment of the loan and accrued interest to date.

Legal Restrictions. When this is the disposition of assurance companies, and when the tendency is so distinctly to grant favors to the public, it seems unfortunate that such conditions should be modified by court decisions. The action of the companies has nevertheless on several occasions been seriously hampered. One obstacle may be mentioned by way of illustration:—A life assurance company had granted a loan on a paid-up policy, and the loan agreement stipulated that if interest on the loan were unpaid the company could treat the policy as surrendered for its customary cash surrender value, the loan to be repaid from such value. The interest was not paid; and, relying on the terms of the contract, the company surrendered the policy and repaid the loan. The assured died shortly afterward, when an action was instituted to recover the full face value of the policy less the indebtedness. The action was sustained on the ground that a policy of assurance should be treated like any other collateral security and sold only in the open market after the usual procedure for foreclosing a mortgage. As this had not been done, a decision was rendered against the company. The effect of this decision, as regards paid-up assurance policies, is to restrict and hamper the companies in granting loans. Companies are generally willing to lend almost the full amount of the net policy value; and, if their only means of protecting themselves against non-payment of interest be that of foreclosure proceedings, the expense of such proceedings is so great that loans on paid-up policies for small amounts may have to be discontinued. Several companies have adopted this course, and it seems a hardship to policy-holders that they should be denied reasonable facilities because of the hampering effect of legal decisions.

The same danger does not exist in the case of loans on policies which are subject to annual premiums, because it is a well recognized principle of law that if the annual premium be not paid the policy will lapse and the risk of the company cease, ex-

cept in so far as the non-forfeiture provisions elsewhere described may serve to continue the assurance.

SURPLUS.

Policies may be divided into two main classes, Participating and Non-participating. Under Non-participating policies it is understood that the sum assured and the premium are definitely guaranteed, and that the assured therefore derives no further benefit from the prosperity of the company with which he is connected than the mere guarantee of his contract. In the case of a Participating policy, however, it is usually provided that a larger premium than that required for the guarantee of the risk be paid. In consideration of this larger premium, an equitable proportion of the profits earned by the company will be placed to the credit of the policy from time to time. The profits allocated to any policy are spoken of as "dividend", "surplus", or "bonus". Most of the surplus is in this manner distributed amongst the policy-holders, dividends to share-holders being usually very small when viewed in relation to the entire surplus. In some companies the dividends to share-holders are strictly limited to a percentage of the share capital, and the usual practice is to distribute by far the greater proportion of the surplus to policy-holders.

Stock Companies. The 1906 laws of New York have laid down a new principle: that companies issuing participating policies must not issue non-participating policies, and also that purely mutual companies cannot issue non-participating policies. Accordingly the stock corporations of New York State have found it necessary to decide whether they will issue participating policies only, or non-participating policies only. The decision of the Legislature is not altogether consistent in this respect, that companies organized in other States than New York are at liberty to continue writing both classes in New York State. Such "foreign corporations" must however keep separate accounts for the two classes of business, but to a certain extent they have an advantage over the home companies.

Sources of Surplus.

The surplus earned by life assurance companies is derived from three principal sources:—

1. The saving in mortality through the careful and judicious selection of healthy lives;
2. Interest earned in excess of the rate assumed for premium and reserve calculations; and
3. Surplus from loadings by reason of the total expenses being less than the total loadings.

There are several minor sources of surplus which are of less consequence than these, such as profit from surrender of policies, from lapse, and from non-participating business. Improvement in value of investments may frequently yield large profits; but these come under the general head of Interest.

Annual and Accumulation Dividends.

The methods of distributing surplus are very varied. In the United States there are two favorite methods, (1) the annual dividend plan, and (2) the accumulation dividend plan. Under the former the surplus found to exist under any policy is allotted in cash each year, and may be taken in reduction of the next premium; or it may sometimes remain on deposit to be carried forward to the credit of the policy, with interest; or it may be applied, if the assured be still healthy, to increase the face value of the policy. Under the accumulation dividend plan the surplus is ascertained periodically under each policy at the end of ten, fifteen, or twenty years, as may be arranged when the policy is taken. By allowing the surplus to accumulate and distributing it only amongst those policies which remain in force, it necessarily follows that a larger amount is apportioned to each policy-holder than could be given if the surplus were taken each year in cash.

There is much to be said in favor of each method. Under the annual dividend plan if the policy-holder should discontinue his policy, or die, he has already obtained his equitable share of the surplus. On the other hand, if he should die early, his death causes so much greater loss to those who continue their policies. Under the accumulation dividend plan, if the policy-holder should

die during the accumulation period advantage of the assurance protection is thereby obtained, whereas, if he survive the accumulation period, the policy, viewed as an investment, is much more satisfactory than if annual dividends had been drawn.

Deferred Dividends Discontinued. A strong wave of public sentiment has passed over the United States against the deferred-dividend system and in favor of an annual distribution of surplus. Several States have enacted laws compelling companies to discontinue the deferred plan. While the distribution of surplus at the end of a long period like twenty years may be objectionable from several standpoints, it would appear as if legislation had run to the other extreme in prescribing an annual division. There are advantages in both methods. Certainly the deferred-dividend system appealed to a selfish instinct in the applicant, namely that he could obtain the profits in one sum for himself on surviving the fixed period. But this very principle added greatly to the popularity of the system, and caused many men, who would have held back from the call of duty, to take out policies with a selfish end in view, and thereby incidentally protect their families from want. Under the usual annual dividend systems it is almost impossible to apportion the surplus accurately, as each policyholder expects his own share at the time his next premium falls due. In the case of policies with anniversaries in January, the share of surplus must be apportioned a month or more before the close of the year. The results must therefore be more or less approximate; indeed if strict accuracy were observed, there would be great fluctuation from year to year and relative discontent amongst policy-holders.

Contribution Plan of Distribution. In allocating the surplus the method used in the United States is that known as the Contribution Plan, which aims at returning to each policy-holder the proportion of surplus which the premiums under the individual policy have contributed. The premiums paid under a policy are placed to its credit and interest thereon is allowed at the rate earned, less the investment expense. To the

debit of the policy there is placed the expense of conducting the business, and the cost of carrying the assurance protection from year to year, both of these being derived from the experience of the company, also the reserve which is necessary at the date of the calculation. The difference between such credits and debits shows the surplus contributed by any individual policy.

Perhaps the above will be better understood if the statement be made in the form of a regular debit and credit account as follows:—

Dr.	Cr.
Expenses of Management,	Premiums Paid Under Policy.
Policy Value Required,	Net Interest Earned, (*i. e.*:—full interest rate less investment expense).
Cost of Assurance, derived from actual experience of Company.	
Balance, being surplus.	

At the second and any subsequent allocation the policy value introduced in the previous calculation would be brought forward as a credit entry—interest thereon being allowed; and the premiums paid would be those relating only to the period elapsed since the preceding allocation.

Other Profits. The above method does not take into consideration the profits from other sources such as surrenders, lapses, annuities, etc. These should be deducted from Expenses, unless it be deemed advisable to apply them towards the formation of a reserve fund for the additional protection of policyholders. The surplus found to have been contributed is then allocated in cash. If the calculation be made annually, it is usual to apply this cash payment towards reduction of the next premium; or, if premiums continue to be paid in full, the surplus is then carried forward to the credit of the policy. Sometimes the surplus is applied to purchase paid-up assurance, thereby increasing the face value of the policy. The usual, but not invariable, custom in America when this course is followed is to have the assured undergo a new medical examination.

British Systems. The British systems of bonus allocation are very different from that above outlined. There are three principal methods employed, known respectively as

1. The Uniform Reversionary Bonus system;
2. The Compound Reversionary Bonus system; and
3. The Contribution Plan.

Uniform Reversionary Bonus. Under the Uniform Reversionary Bonus system, when the surplus has been ascertained by means of a valuation frequently conducted only once in each five years, it is all, or nearly all, distributed amongst the policy-holders in the form of additions to the face values of their policies. These additions are made without evidence of health, and at a uniform rate for each premium paid or for each year in force. A policy for £1,000 frequently has its face value increased at the rate of about £15 each year, or even at higher rates, by means of bonus distributions. The regular distributions generally take place each five years; and, in the interim, temporary additions at a smaller rate are made, these "interim bonuses" being superseded when the regular bonus is declared. Under the uniform system the addition is calculated on the face value of the policy only.

Compound Reversionary Bonus. Under the Compound Reversionary Bonus system the addition to the face value of the policy is calculated not only upon the original amount of the policy, but on the total amount as increased by the previous bonuses. The difference between the Uniform and the Compound Reversionary system is analogous to that between simple and compound interest.

The effect of either system is to increase continually the sums assured in force. Moreover, as the addition to the face value is at a uniform rate irrespective of age, it follows that the cash equivalent of the addition is much greater in the case of old lives than of young. Under the Compound system the policy-holders who have been longest assured get the largest bonuses. The

system encourages persistence in maintaining policies; and it may be added that it is very popular.

British Contribution Plan. The Contribution Plan is applied in Great Britain in an entirely different way from the American method. The total surplus is ascertained at periodic intervals, usually five years, in the same way as when other forms of distribution are in vogue. This surplus is then "analyzed", and the amount earned from different sources ascertained. The total surplus is then set apart in two principal divisions—

1. Profit from surplus interest; and
2. Profit from loading,

these being the two main sections of surplus in Great Britain.

The profit from surplus interest is allocated according to the reserve value of the policy, and the profit from loading according to the loadings paid under the policy since the date of the previous distribution, these being the factors which have enabled the company to earn the two classes of surplus respectively.

Under the Uniform or Compound Reversionary Bonus systems the premiums should be specially calculated to provide that form of bonus which it is intended to pay; otherwise there is likely to be unfairness as between the entrants at different ages. When premiums have not been so calculated, the Contribution Plan, as above outlined, has the effect of roughly adjusting the equities as between the policy-holders of different ages and durations.

Death Claims.

The principal object of a life assurance company is to pay the sums assured in event of death. Reference was made to the unfortunate use of the word "reserves", as policy values are frequently designated; and in a similar way the word "losses", so often applied to the claims by death, conveys an entirely erroneous impression. Death Claims are not "losses"; they are pay-

ments which are anticipated and for which provision is made in advance, and very seldom indeed are they in excess of the sums for which provision is made. In most cases *favorable* mortality is a source of great gain to a life assurance company.

When the life assured under a policy dies, intimation of the death should be sent to the company or to one of its branch offices, and instructions are then given by the Company as to the steps necessary to prove the claim and obtain payment of the sum assured. The procedure is simple, and legal advice on the part of an honest claimant is in most cases an unnecessary expense. There are two essential conditions which must be fulfilled before payment can be obtained. The *first* is that the life assured must really be dead; and the *second* that the person claiming has the right to receive the money.

Proofs of Death. The practice of the Companies requires that several forms be completed to prove that death has actually taken place. The claimant's own statement gives the name and the capacity in which the claim is being made, details of the deceased person, such as age, residence, occupation, etc., which are necessary to make sure that the particulars agree with those submitted with the application for the policy—in other words, to make certain that the deceased is the person on whom the policy was issued. The date and cause of death are also required, and other questions are frequently asked relating to policies held by the deceased in other companies, etc.

A statement from the physician who attended the life assured is necessary, and this deals principally with the cause of death and the length of time during which the deceased person suffered from illness. A reference to any other physician who may have attended the life assured on previous occasions is desirable.

It is usual to obtain also a certificate from a personal friend certifying as to the identity of the deceased, and possibly also a certificate from a clergyman, both of which are confirmatory of the claimant's statement and intended to elicit information which would distinguish a fraudulent claim from those which are hon-

est. An undertaker's certificate relating to the interment may also be procured; and if a coroner's inquest were held, a copy of the verdict should be furnished.

Incidentally it is sometimes necessary to have evidence that the age of the life assured had been correctly stated when the policy was taken. In America the usual custom is to accept the statements when the claim is being made if they agree with those given in the original application, although further evidence could legitimately be required. If any disagreement appears, then further inquiry is made. In Great Britain it is usual to call for more definite proof, such as a certificate of birth, an extract from family records, or a declaration by some person who had known deceased for a long time. One of the important documents in proving a death in that country is the official Registrar's Certificate.

Options at Settlement. The most usual form of policy provides for the payment of the sum assured in cash at death or at the maturity of an Endowment Policy; but frequently the beneficiary has the right to choose other forms of settlement, such as having the sum assured payable in instalments during a specified number of years, usually twenty years. Or the sum assured may be left on deposit with the company during the life-time of the beneficiary, a fixed rate of interest in the nature of a life annuity being paid each year. The rate of interest guaranteed in the policy is usually 3%, so that beneficiaries are not likely to take much advantage of this provision. Sometimes the offer is made to pay an annuity during the life-time of the beneficiary at rates fixed in advance, depending on the attained age at time of settlement. And lastly, another favorite mode of settlement is that by Continuous Instalments, which method guarantees a fixed annuity for a definite number of years, like settlement in instalments, and continues the annual payment thereafter so long as the beneficiary may survive. A table of the rates, depending on the attained age when the option is exercised, is usually incorporated in the policy.

Capacity of Claimant. The persons who have the proper right to receive payment of the sum assured are most usually represented by one or other of the following:—

1. Beneficiary;
2. Administrator;
3. Executor under a will;
4. Assignee.

If a specific beneficiary is named in a policy, and if such beneficiary is of full age and able to execute the necessary papers, the case is the simplest of all. The second and third cases are very similar, and in each of them the appointment of the executor or administrator must be duly made by the proper court, a certified copy being furnished to the assurance company. When there is no will it is usual for the court to appoint some near relative to administer the estate of the deceased; and when there is a will, the executor or executors named in it must have their title confirmed by the court. The court procedure in such cases has been made exceedingly simple, so that the assistance of a lawyer in procuring the necessary papers is seldom necessary.

In the case of a policy which has been assigned, many questions may arise as to the validity of the assignment and the rights of the assignee. In some states it is held that a policy of assurance can only be assigned to those who already have an interest in the life assured, or who are within certain admitted degrees of kinship; but the more businesslike principle seems to be gaining ground that a policy may be bought and sold like any other piece of personal property. If the assignee under a policy holds his title merely to secure a loan, he must pay over the balance of the policy moneys, after satisfying the debt, to the estate of the assured. It is sufficient here to mention such questions without entering at length into the more unusual cases.

CHAPTER XIII.

State Supervision and Annual Statements.

The business of life assurance is unlike any ordinary commercial business. In one sense it is purely commercial in that it consists in buying and selling, the commodity being policies of assurance. But the purchase price is paid by instalments over a number of years, and the payment, made by the company in exchange for that purchase price, is deferred for a long period of years. In the early days of life assurance this peculiar condition led to gross frauds upon the public by irresponsible companies, who, after receiving and dissipating premiums, were in no position to pay the sums assured when the policies matured. Accordingly it became necessary to regulate the business and protect the public against fraudulent companies; otherwise such companies had the appearance of prosperity for many years without any sound foundation.

Supervision in General. In nearly all countries where life assurance is practiced, some Government regulations have been imposed, making it necessary that this business be carried on under stricter rules than those which govern ordinary commercial enterprises. The first and most obvious precaution is that the companies must publish statements of their affairs at periodic intervals, and the second that they should make accurate calculations showing the reserve liability under the policies necessary to secure payment of the sums assured. The great principle of Government supervision therefore has been to insist upon the publication of annual accounts and the rendering

of valuation statements at such intervals as may be determined. In other respects, the management is frequently left unhampered by regulation on the ground that competition will cause each company to render the best possible returns to its patrons and policy-holders.

This mild and necessary form of supervision has been replaced in many instances by careful regulation of the business itself, on the ground that life assurance companies constitute what might almost be called a "public trust". In most countries a standard of solvency is prescribed, and if a life assurance company does not have assets of sufficient value to meet its ascertained liabilities by the statutory standard, it must then cease to do business. In fixing a standard of solvency, it is necessary to prescribe (1) the rate of interest to be used in the calculations, (2) the table of mortality, and (3) the method of valuation, that is, whether by net premiums, by modified gross premiums, or by the actual gross premiums themselves.

The variation arising from the use of different mortality tables in temperate countries is not very great, but very different results are obtained according to the rates of interest which may be prescribed. In the United States these rates of interest generally vary from 3½% to 4½% (see page 53), and in most States companies are *permitted* to use a stricter mode of valuation if they prefer to do so. In New York and Massachusetts the companies have to value all business written after 1900 at 3½% although the average rate of interest earned is well over 4%.

Strict Regulations. The principles of supervision have been made much more strict in the recent laws of New York. The methods of conducting the business have been placed within definite limits in such matters as the expense which can be incurred for writing new business, the kinds of policies which companies may issue, and even the amount of business which a large company may write. The forms of policy contracts for use in New York State have been put forward as "Standard" Policy forms (see Chapter II.). Companies may not issue both

participating and non-participating policies, and when participating policies are issued they must provide for the annual distribution of surplus. Mutual companies cannot issue non-participating contracts, except annuities. The method of voting for directors in mutual companies has been completely changed, and policy-holders in stock companies are eligible as directors whether stockholders or not. Many other regulations have been introduced, dealing with the management; and these are based upon principles of State Regulation which had never before been tried.

These regulations have not been generally adopted by other Legislatures throughout the United States where the same questions have been considered. In particular, a commission appointed by the Governor of Massachusetts reported to the Legislature that a limitation as to the amount of expenses is inadvisable, that the restriction of investments would be likely to prove harmful, and that they would not recommend for adoption laws similar to those in New York, permitting policy-holders to vote for directors by mail and necessitating the publication of lists of policy-holders of mutual companies.

Efforts at Uniformity. A strong effort has of late been made to secure greater uniformity in the laws regulating life assurance, but this effort seems to have received a check by the radical changes hurried through the New York Legislature in 1906, and which, as above indicated, other States will not follow. The Insurance Commissioners of the different States meet annually in convention, and these meetings tend towards harmony, because questions on the welfare of life assurance throughout the whole country are then discussed. Opinions are freely exchanged and deference shown to the wishes of the majority.

Particular reference may be made to the work of securing uniformity in the statement blanks which the companies have to use in making their annual returns to Insurance Departments. Each company has to file with the Commissioner of each State in which business is being done, an annual report, showing the

receipts and disbursements of the company during the preceding year, its assets and liabilities itemized, particulars of the new business written during the year, the business terminated, and the amount remaining in force, all by classes of policies.

Uniform Statement Blank. Each State had formerly its own schedule of questions, and much confusion arose as to the manner in which certain questions should be answered. In many cases the difficulty was caused by different wording of a question intended to elicit the same information. The convention of Insurance Commissioners has now recommended for general use a uniform blank, but of course the convention has no power to insist on this being used. It has not been adopted by all the States, but most of them use it, and much trouble is avoided thereby, while the public are not misled but are better informed, finding that the reports to different States agree with one another. Without entering into all the details of the uniform blank, a brief outline of the principal items and information elicited is as follows:—

ANNUAL STATEMENT OF THE .... COMPANY.

Ledger Assets at beginning of year, $....

INCOME.		DISBURSEMENTS.	
First year's premiums,	$....	Death claims and matured endowments,	$....
Consideration for annuities,	....	Annuities paid,	....
Renewal premiums,	....	Surrender values paid,	....
Interests and rents,	....	Dividends paid,	....
Miscellaneous receipts,	....	Commissions to agents,	....
		Salaries,	....
		Medical fees,	....
		Rent, advertising, printing, etc.,	....
		Taxes, licenses and fees,	....
		Other disbursements,	....
		Total Disbursements,	$....
		Balance, being Ledger Assets at end of year,	$....
	$		

LEDGER ASSETS.

Book Value of real estate,	$....
Mortgage loans,	....
Loans to policy-holders,	....
Book Value of stocks & bonds,	....
Cash on deposit and in hand,	....
Agents' Balances and other assets,	....
Total Ledger Assets,	$

NON-LEDGER ASSETS.

Interest and rents accrued and unpaid,	$....
Market Value of investments over Book Value,	....
Premiums outstanding,	....
	$....
Deduct assets not admitted, such as office furniture, stationery and supplies, agents' balances, loans on personal security, bills, etc.,	$....

LIABILITIES.

Reserve for policies and annuities,	$....
Claims for death losses unpaid or in process of adjustment,	....
Premiums paid in advance,	....
Commissions payable to agents on outstanding premiums,	....
Dividends apportioned to policy-holders,	....
Other liabilities,	....
Capital stock, if any,	....
Unassigned funds, (surplus),	....

Following the above statements there is an "Exhibit of Policies" showing the business in force by kinds at the beginning of the year, the new policies written, those terminated and the business remaining in force at the close of the year. All this information is given by sums assured and with further details as to kinds of policies written, and modes of termination. Detailed information is also included in regard to the assets, with schedules of individual investments in real estate, in bonds, mortgages, etc.

In future much other important information will be furnished under this section of the blank, such as:—(1) detailed account of all legal expenses; (2) the names of officers and directors with compensation to each; (3) list of death claims resisted or compromised with reasons for special treatment; (4) a complete statement of the profits and losses on the business transacted during the year, with the sources of such gains and losses; (5)

particulars of dividends on policies, annual or deferred; and (6) particulars of reserve and surplus funds and the purpose for which these are held.

Cash Accounts. Receipts and disbursements are required in the uniform blank on a cash basis. The income which *falls due* during the year, and the liabilities *incurred*, are not required, but only the cash receipts and the cash payments. An attempt is made to overcome the awkwardness of this method by adjusting the assets and the liabilities, which show certain items of "non-ledger assets", as well as outstanding liabilities; but this attempt is only partly successful, and it complicates the form of account. To find the true income of a company it is necessary not only to refer to the account of receipts, but also to refer back to various items of assets in the previous year's statement, find the difference which has arisen in these items during the year and treat this difference as an addition to, or deduction from, the statement of receipts so as to get the true income for the year.

For illustration, the true Premium revenue of the year 1905 in a company can only be obtained as follows:—

Add	(1) Cash Premium Receipts for 1905		$1,000,000
	(2) Premiums Outstanding taken from "Assets" on 31st Dec. 1905		100,000
	(3) Premiums Paid in Advance taken from "Liabilities" on 31st Dec. 1904		10,000
			$1,110,000
Deduct	(1) Premiums Outstanding, taken from "Assets" on 31st Dec. 1904	$90,000	
	(2) Premiums Paid in Advance taken from "Liabilities" on 31st Dec. 1905	15,000	105,000
True Premium Revenue for Year 1905			$1,005,000

Simplify Statements. As the object of publishing accounts is to inform the public, it is desirable to give accurate information in the simplest form. Experts on accounts, such as Officials of Companies, State Commissioners, and their employees, may be able to get true results from the present forms, but not so the public. It would be much better that the experts should have the trouble of bringing the items together in a revenue account, thus simplifying the form of statement put before the public. It should not be left to the public to pick some figures from one account and some from another before they can get a clear conception of the true income and expenditure. The Gain and Loss Exhibit, afterwards referred to, is prepared on the basis of true revenue accounts, and is to this extent entirely different in character from the usual annual statements of a company.

Differences in Annual Statements. The causes of the differences between the reports of different States at the present time may be mentioned very briefly:—

(1) Some States insist that those policies be ignored entirely on which the first premium has not yet been paid, although the policy had been duly issued, though perhaps not delivered before 31st December. This causes a difference both in Assets and Liabilities.

(2) The basis of reserve valuation differs considerably, and some States refuse to accept any other than their own basis.

(3) Assets of the nature of Agents' Balances, Office Furniture, and Personal Loans are occasionally admitted, although the general practice is to treat these as if they did not exist.

In consequence of these, and other differences in treatment, it sometimes happens that the same Company may appear to have Assets varying by hundreds of thousands of dollars in several State Insurance Reports.

COMPARATIVE STATEMENTS.

In judging the advantages offered to policy-holders, the characteristics of a company should be taken into consideration, as represented by (1) its economy of management, (2) the interest rate earned, and (3) its mortality experience. In addition to these the premium rate and the basis of valuation are important, but if a company is capably and economically administered, the surplus returned to policy-holders bears a proper relation to the premiums. A company charging lower premiums cannot be expected to pay the same dividends as a company charging higher premiums, even although the two are equally well administered in the interests of the policy-holders.

Dividends being paid by a company are not always an index of the future, because some companies have distributed more surplus than they have been earning, thus using up the accumulations brought forward from earlier years. The earning power of a company may be different from the dividend results; and the questions of Economy, Interest, and Mortality are more important than dividends declared. Especially is this the case when comparisons are made of deferred dividends. The results of twenty years already past do not give a good index or estimate of the results to be obtained twenty years hence. Business conditions change completely even within a period of ten years, while changes in management and other disturbing elements make the forecasting of results with accuracy almost an impossibility.

It has become common to examine the condition of different companies by means of comparative statements, and a general view of these enables an expert to form a good opinion. But to the inexperienced the statements are often misleading and it is therefore desirable to point out the more common mistakes which are made.

Expense Comparison. Expenses of management are generally compared with the premium income of the company; sometimes the comparison is made with the total income including interest on investments. Neither is of much value, although

the first is better than the second. A large part of the expense of conducting a company is caused by the new business, and any company doing a large proportion of new business is bound to show a high expense ratio, when compared with another company where the proportions are reversed and the new business bears a small ratio to the old. For example, if the new premiums of a company amounted to $500,000 and the renewal premiums amounted to $5,000,000, an expense rate of 100% of new premiums and 10% of renewal premiums would give a total expenditure of $1,000,000, representing about 18% of the total premiums. On the other hand, if the company were to transact the same amount of new business, but had a renewal premium income of only $2,000,000, the expenses would be $700,000, that is, $500,000 for new premiums and $200,000 for renewals, and this would be no less than 28% of the premium income. Accordingly, although the two companies are conducted on exactly the same basis of expense, the one has the *appearance* of being much more economically managed than the other, if the only comparison be that of total expenses to total premiums.

The same condition is exaggerated yet further by comparing the expenses with the total income, because old companies with a large ratio of renewal income have also large assets, the interest on which increases the income, whereas younger companies with a small ratio of renewal income have smaller assets and therefore less interest income, so that they appear at a great disadvantage if erroneous comparisons are made in this way.

Better Expense Comparison. One of the best methods of testing the economy of a life assurance company is to assume that the expense of carrying the old business bears a fixed proportion to the expense of writing new business. The system generally approved is that the writing of new business costs ten times as much as the care of the old, this ratio being used in the foregoing illustration. The adoption of this ratio is arbitrary, but it is reasonably fair, and the system, though still imperfect, is better than the old plan of comparing total expenses with total premiums. The result in individual cases would be that the com-

pany might show 95% of new premiums, and 9.5% of renewals; another more economical company would show the expenses to be 80% of new premiums and 8% of renewals, and so forth. This method is incomplete in that it assumes that all expenses are percentages of the premiums while, as indicated in Chapter IX., expenses are partly constant in relation to the sums assured.

Interest Comparison. The best method of computing the rate of interest earned by a life assurance company is given in Chapter IV., but most frequently the published statements give merely the ratio of the interest paid to the mean assets, and in many of them the figures are incorrectly taken from the returns. In addition to the interest receipts in the cash account above mentioned, the difference between the accrued interest at the end of the year and of the preceding year should be added to or subtracted from the cash interest and rents received. Certain forms of expenses shown in the disbursements, such as repairs and expenses on real estate, should be deducted so as to get the net yield of investments. There may also be added to the interest of the year the difference between (1) the profit on the sale of securities plus any increase in market values and (2) the losses on sales of securities, plus any decrease in market values. The true income for the year may thus be accurately ascertained and divided by the mean of the assets as of the beginning and end of the year as shown by the formula on page 55. But published comparisons seldom, if ever, take these adjustments into consideration, with the result that erroneous and almost useless comparisons are put forth.

Mortality Comparison. The published comparisons of mortality are in like manner more than doubtful. Frequently the death claims paid are compared with the amount of assurance outstanding and the result published as "death-rate-per-thousand". This comparison takes no account of the ages of the lives assured, so that a company with many old risks *must* show a heavy death-rate when compared with a young company where the lives are on the average about middle age or younger. Yet

the higher death-rate in the first, may yield greater mortality profit to the company than the lower death-rate in the second; or *vice versa.* In other words such a comparison, by itself, tells almost nothing. Then again, any company with a large new business in proportion to the old business in force will naturally show a low mortality rate on account of the effect of selection referred to in Chapter III. A good comparison of mortality is almost impossible from any published figures. Doubtless the best can be made by means of the Gain and Loss Exhibit, if the figures in that exhibit are accurately and carefully prepared; but this is not always the case, and moreover no consideration is given to the selection above mentioned.

GAIN AND LOSS EXHIBIT.

The only fair means of making comparisons of interest and mortality in this country is by reference to the Gain and Loss Exhibit, a form of statement which has been called for by various State Departments since the year 1893. This exhibit purports to analyze the accounts of a company for any particular year, and show the sources of surplus. The more important items in the exhibit on one side of the account, are

(1) Surplus brought forward,
(2) Loadings on premiums for the year,
(3) Interest income for the year,
(4) Expected mortality,
(5) Reserve released on surrendered and lapsed policies, and
(6) Miscellaneous credits.

On the other side of the account there are shown

(1) Total management expenses,
(2) Interest required at the valuation rate to maintain the reserve,
(3) Actual net mortality,
(4) Values paid and applied on the surrender of policies,
(5) Dividends to policy-holders,
(6) Miscellaneous debits, and
(7) Surplus carried forward.

While each item in this Exhibit is supposed to be separately computed, the statement is prepared with the understanding that it should balance. But there are several items which are not included, and an exact balance is therefore almost impossible. Most frequently one of the items is "forced" to effect the balance required.

By comparing the loadings with the expenses of management the profit or loss from loading is obtained. By comparing the expected mortality with the actual mortality the gain or loss from that source is obtained, and so on for other items. But the comparison of loadings with expenses is incomplete because if a company carries 3% reserves and charges average premiums, it has the appearance of spending a larger part of its loadings than another company charging the same premiums and incurring the same expenses, yet carrying the smaller 3½% reserves. This arises from the fact that the 3% net premiums are larger than the 3½% net premiums, so that there is a smaller margin of loading in the former. Accordingly, the company which carries the larger reserves has the appearance of being more extravagantly managed because *a larger proportion of its loading is spent;* its expenses may actually be less than those incurred by the company valuing on a 3½% basis. Such a comparison is therefore unfair unless other facts are taken into account.

The difficulty in making up this Exhibit accurately has resulted in Actuaries approximating the figures, and such approximations have often been unsatisfactory. If the figures used were entirely reliable, the statement would be a most valuable one for comparative purposes. The Exhibit does not make provision for certain forms of changes in reserves, nor for credits on expenses, and it occasionally happens that through misunderstanding or ignorance on the part of the person compiling the figures, an important item of gain or loss is placed in the wrong section of the statement, thereby distorting the results, and making the entire statement poor for comparative purposes. If a change be made in the basis of reserve under some class of policies, such as a reduction in the rate of interest, this disturbs the statement and leads to erroneous conclusions. Such changes should be carefully dealt with, and reported as special items of gain or loss

as the case may be. It requires very considerable skill to prepare the statement properly, indeed it is almost impossible to compute the figures with complete accuracy.

Scientific Principles. Nevertheless the Gain and Loss Exhibit is a scientific document, correct in theory and valuable to the expert, although most misleading to the average student of Life Assurance. It recognizes the proper principles of bookkeeping and deals with the true revenue and expenditure of the year, to that extent giving more valuable information than is contained in the cash accounts of actual receipts and disbursements. It is to be hoped that the advantage of this course will impress itself on the various State Insurance Officials, so that they may recommend the extension of the principle to cover all annual accounts required from companies. This would result in the statements being much simpler and more easily understood by the public. They would also be drawn on the scientific lines recommended by the best expert accountants.

APPENDIX.

Principles of Notation.

The notation employed in actuarial science follows fixed rules, and if hese are understood the expressions can be readily interpreted. So far as possible the initial letters of words have been employed as the general basis, such as i for interest; l for living; d for dying; p for probability of living; A for assurance; and a for annuity. In some cases it is not possible to use initial letters, and others have to be arbitrarily chosen, such as q, which represents the probability of dying, being the next letter to p, the probability of living. Further explanations are given in the text as occasion arises.

These important central letters are modified by smaller letters or symbols to the left or right, and slightly below the basic letter; frequently, also, by letters in the upper right or left corners, and by marks placed above. The smaller letters, or subscripts, indicate the age of the life or lives under consideration, and may limit the time during which the contract would remain in force. The last letters of the alphabet, v, w, x, y, and z, are usually employed to indicate the ages of the lives under consideration. When one of these letters is placed in small brackets it relates to a person of that age; thus (x) means "a person aged x", (y) "a person aged y", etc.

l_x is the number living at age x.
l_{x+n} is the number living at age $x+n$.
p_x is the probability of living one year at age x.
p_{x+n} is the probability of living one year at age $x+n$.
p_{xy} is the probability that both (x) and (y) will live one year.

Note. In the above probabilities 1 is *understood* without being expressed as a subscript to the left of the basic letter. It is the time over which the probability is computed.

The number of years over which a probability or benefit would extend is generally shown by the use of a letter, such as n, m, or r, either before or after the basic letter, as follows:—

${}_np_x$ is the probability that a life aged x will live n years.

${}_{|n}a_x$ or $a_{x\overline{n|}}$ represent an annuity on (x) to run for n years, provided he live so long.

$A_{x\overline{n|}}$ is the value of the sum assured, payable either at the death of (x) or on the expiry of n years. It therefore indicates the present value of an endowment assurance.

$a_{\overline{n|}}$ is an annuity running for n years certain, irrespective of any life.

Important

When the last survivor of two or more lives is in question, a horizontal bar is placed above the letters representing their ages, in the manner following:—

$_np_{\overline{xy}}$ is the probability that the survivor of (x) and (y) will live for n years.

$a_{\overline{xyz}}$ is an annuity payable during the joint lives of (x), (y), and (z), and continuing until the death of the last survivor.

In order to indicate the survivorship of one life after another, such as the value of a benefit to (x) after the death of (y), an upright bar is used, separating the two lives as follows:—

$a_{y|x}$ is the present value of an annuity payable during the continued lifetime of (x) after the death of (y).

$Pa_{y|x}$ is the annual premium for the same benefit.

To indicate the order of death, a small number is used either below or above the letter representing the age. If the number is given above the letter, it also shows that the benefit is payable on the fulfilling of this condition.

$A_{\overset{1}{x}\,y}$ is the single premium for an assurance of 1 payable on the death of (x) if he die first, i. e:—in the lifetime of (y).

$A_{\overset{3}{x}\,\underset{2}{y}\,\underset{1}{z}}$ gives the value of an assurance payable on the death of (x) if he die third, provided that (z) shall have died first and (y) second.

To indicate a deferred term, an upright bar is used to the left of the central letter.

$_{n|}a_x$ is an annuity deferred n years, and payable during the subsequent lifetime of (x).

The same function *could* be expressed under the rule given in a preceding paragraph, thus:—

$a_{\overline{n|}|x}$, which equals $_{n|}a_x$, the more usual form.

To indicate the subdivision of a year, a small letter in brackets is generally used at the upper right corner.

$a_x^{(m)}$ is an annuity to (x) payable m times a year.

$P_x^{(m)}$ is the annual premium, payable m times a year, for an assurance on (x).

To indicate a continuous or momentary benefit, a small bar is generally placed above the basic letter.

$\bar{A}_x$ is the value of an Assurance payable at the moment of death of (x).

$\bar{a}_x$ is an annuity payable at infinitesimal intervals.

To indicate that a benefit is complete, a small circle is placed above the basic letter.

$\overset{\circ}{a}_x$ represents an annuity payable once each year, but with a proportionate payment from the last periodic payment to the date of death, thereby making it complete.

The notation here outlined is capable of almost indefinite extension to meet varying circumstances; and, by the use of symbols, ideas can be expressed more accurately as well as much more briefly. They have a mathematical significance and rigid meaning which cannot be misunderstood, whereas the slightest turn of a phrase sometimes gives a different shade of meaning to a sentence expressed in words. It will be observed that one rule occasionally overlaps another, so that the same ideas can be expressed in two ways; but this is no disadvantage when the system is understood.

TABLE I.

Amount to which 1 will accumulate:— $(1+i)^n$

NO. OF YEARS.	RATE OF INTEREST.		
	3½%	1¾%	⅞%
n	$i=.035$	$i=.0175$	$i=.00875$
1	1.035000	1.017500	1.008750
2	1.071225	1.035306	1.017577
3	1.108718	1.053424	1.026480
4	1.147523	1.071859	1.035462
5	1.187686	1.090617	1.044522
6	1.229255	1.109702	1.053662
7	1.272279	1.129122	1.062881
8	1.316809	1.148882	1.072182
9	1.362897	1.168987	1.081563
10	1.410599	1.189444	1.091027
11	1.459970	1.210260	1.100573
12	1.511069	1.231439	1.110203
13	1.563956	1.252990	1.119918
14	1.618695	1.274917	1.129717
15	1.675349	1.297228	1.139602
16	1.733986	1.319929	1.149574
17	1.794676	1.343028	1.159632
18	1.857489	1.366531	1.169779
19	1.922501	1.390445	1.180015
20	1.989789	1.414778	1.190340
21	2.059431	1.439537	1.200755
22	2.131512	1.464729	1.211262
23	2.206114	1.490361	1.221860
24	2.283328	1.516443	1.232552
25	2.363245	1.542981	1.243337
26	2.445959	1.569983	1.254216
27	2.531567	1.597457	1.265190
28	2.620172	1.625413	1.276261
29	2.711878	1.653858	1.287428
30	2.806794	1.682800	1.298693
31	2.905031	1.712249	1.310056
32	3.006708	1.742213	1.321519
33	3.111942	1.772702	1.333083
34	3.220860	1.803725	1.344747
35	3.333590	1.835290	1.356514
36	3.450266	1.867407	1.368383
37	3.571025	1.900087	1.380357
38	3.696011	1.933338	1.392435
39	3 825372	1.967172	1.404618
40	3.959260	2.001597	1.416909
41	4.097834	2.036625	1.429307
42	4.241258	2.072266	1.441813
43	4.389702	2.108531	1.454429
44	4.543342	2.145430	1.467155
45	4.702359	2.182975	1.479993
46	4.866941	2.221177	1.492943
47	5.037284	2.260048	1.506006
48	5.213589	2.299599	1.519184
49	5.396065	2.339842	1.532477
50	5.584927	2.380789	1.545886

TABLE II.

Present value of 1 :— $v^n = \frac{1}{(1+i)}$

NO. OF YEARS.	RATE OF INTEREST.		
	3½%	1¾%	⅞%
n	$i = .035$	$i = .0175$	$i = .00875$
1	0.966184	0.982801	0.991326
2	0.933511	0.965898	0.982727
3	0.901943	0.949285	0.974203
4	0.871442	0.932959	0.965752
5	0.841973	0.916913	0.957375
6	0.813501	0.901143	0.949071
7	0.785991	0.885644	0.940839
8	0.759412	0.870412	0.932678
9	0.733731	0.855441	0.924588
10	0 708919	0.840729	0.916568
11	0.684946	0.826269	0.908617
12	0.661783	0.812058	0.900736
13	0.639404	0.798091	0.892923
14	0.617782	0.784365	0.885177
15	0.596891	0.770875	0.877499
16	0.576706	0.757616	0.869888
17	0.557204	0.744586	0.862342
18	0.538361	0.731780	0.854862
19	0.520156	0.719194	0.847447
20	0.502566	0.706825	0.840096
21	0.485571	0.694668	0.832809
22	0.469151	0.682720	0.825585
23	0.453286	0.670978	0.818424
24	0.437957	0.659438	0.811325
25	0.423147	0.648096	0.804287
26	0.408838	0.636950	0.797311
27	0.395012	0.625995	0.790395
28	0.381654	0.615228	0.783539
29	0.368748	0.604647	0.776743
30	0.356278	0.594248	0.770005
31	0.344230	0.584027	0.763326
32	0.332590	0.573982	0.756705
33	0.321343	0.564111	0.750141
34	0.310476	0.554408	0.743634
35	0.299977	0.544873	0.737184
36	0.289833	0.535502	0.730789
37	0.280032	0.526292	0.724451
38	0.270562	0.517240	0.718167
39	0.261413	0.508344	0.711937
40	0.252572	0.499601	0.705762
41	0.244031	0.491008	0.699640
42	0.235779	0.482563	0.693571
43	0.227806	0.474264	0.687555
44	0.220102	0.466107	0.681591
45	0.212659	0.458090	0.675679
46	0.205468	0.450212	0.669818
47	0.198520	0.442469	0.664008
48	0.191806	0.434858	0.658248
49	0.185320	0.427379	0.652539
50	0.179053	0.420029	0.646878

TABLE III.

The accumulated amount of 1 per annum *in advance*:—*

$$(1+i)\, S_{\overline{n|}} = \frac{(1+i)^{n+1}-1}{i} - 1$$

NO. OF YEARS.	RATE OF INTEREST.		
	3½%	1¾%	⅞%
n	$i = .035$	$i = .0175$	$i = .00875$
1	1.035000	1.017500	1.008750
2	2.106225	2.052806	2.026327
3	3.214943	3.106230	3.052807
4	4.362466	4.178089	4.088269
5	5.550152	5.268706	5.132791
6	6.779408	6.378408	6.186453
7	8.051687	7.507530	7.249335
8	9.368496	8.656412	8.321516
9	10.731393	9.825399	9.403080
10	12.141992	11.014844	10.494107
11	13.601962	12.225104	11.594680
12	15.113030	13.456543	12.704884
13	16.676986	14.709533	13.824801
14	18.295681	15.984449	14.954518
15	19.971030	17.281677	16.094120
16	21.705016	18.601607	17.243694
17	23.499691	19.944635	18.403326
18	25.357181	21.311166	19.573105
19	27.279682	22.701611	20.753120
20	29.269471	24.116389	21.943460
21	31.328902	25.555926	23.144215
22	33.460414	27.020655	24.355477
23	35.666528	28.511016	25.577337
24	37.949857	30.027459	26.809889
25	40.313102	31.570440	28.053226
26	42.759060	33.140422	29.307441
27	45.290627	34.737880	30.572631
28	47.910799	36.363293	31.848892
29	50.622677	38.017150	33.136320
30	53.429471	39.699950	34.435012
31	56.334502	41.412200	35.745069
32	59.341210	43.154413	37.066588
33	62.453152	44.927115	38.399671
34	65.674013	46.730840	39.744418
35	69.007603	48.566129	41.100932
36	72.457869	50.433537	42.469315
37	76.028895	52.333624	43.849671
38	79.724906	54.266962	45.242106
39	83.550278	56.234134	46.646724
40	87.509537	58.235731	48.063633
41	91.607371	60.272357	49.492940
42	95.848629	62.344623	50.934753
43	100.238331	64.453154	52.389182
44	104.781673	66.598584	53.856338
45	109.484031	68.781559	55.336331
46	114.350973	71.002736	56.829273
47	119.388257	73.262784	58.335280
48	124.601846	75.562383	59.854463
49	129.997910	77.902225	61.386940
50	135.582837	80.283014	62.932826

* This form of table is more frequently used in practice than the simple table of $S_{\overline{n|}}$

TABLE IV.

The present value of 1 per annum:—

$$a_{\overline{n}|} = \frac{1-v^n}{i}$$

NO. OF YEARS.	RATE OF INTEREST.		
	3½%	1¾%	⅞%
n	$i = .035$	$i = .0175$	$i .00875$
1	0.966184	0.982801	0.991326
2	1.899694	1.948699	1.974053
3	2.801637	2.897984	2.948256
4	3.673079	3.830943	3.914008
5	4.515052	4.747855	4.871384
6	5.328553	5.648998	5.820455
7	6.114544	6.534641	6.761293
8	6.873956	7.405053	7.693971
9	7.607687	8.260494	8.618559
10	8.316605	9.101223	9.535126
11	9.001551	9.927492	10.443743
12	9.663334	10.739550	11.344479
13	10.302738	11.537641	12.237402
14	10.920520	12.322006	13.122579
15	11.517411	13.092880	14.000079
16	12.094117	13.850497	14.869967
17	12.651321	14.595083	15.732309
18	13.189682	15.326863	16.587171
19	13.709837	16.046057	17.434618
20	14.212403	16.752881	18.274714
21	14.697974	17.447549	19.107524
22	15.167125	18.130269	19.933109
23	15.620410	18.801248	20.751534
24	16.058368	19.460686	21.562858
25	16.481515	20.108782	22.367145
26	16.890352	20.745732	23.164456
27	17.285365	21.371726	23.954852
28	17.667019	21.986955	24.738391
29	18.035767	22.591602	25.515133
30	18.392045	23.185849	26.285138
31	18.736276	23.769877	27.048464
32	19.068865	24.343859	27.805169
33	19.390208	24.907970	28.555310
34	19.700684	25.462378	29.298944
35	20.000661	26.007251	30.036128
36	20.290494	26.542753	30.766018
37	20.570525	27.069045	31.491368
38	20.841087	27.586285	32.209535
39	21.102500	28.094629	32.921472
40	21.355072	28.594230	33.627234
41	21.599104	29.085238	34.326873
42	21.834883	29.567801	35.020444
43	22.062689	30.042065	35.707999
44	22.282791	30.508172	36.389591
45	22.495450	30.966262	37.065269
46	22.700918	31.416474	37.735087
47	22.899438	31.858943	38.399095
48	23.091244	32.293801	39.057344
49	23.276564	32.721181	39.709882
50	23.455618	33.141209	40.356760

TABLE V.

The annuity which 1 will purchase:—

$$\frac{1}{a_{\overline{n|}}} = \frac{i}{1-v^n}$$

NO. OF YEARS. n	RATE OF INTEREST. 3½% $i = .035$	1¾% $i = .0175$	⅞% $i = .00875$
1	1.035000	1.017500	1.008750
2	0.526400	0.513163	0.506572
3	.356934	.345067	.339184
4	.272251	.261032	.255493
5	.221481	.210621	.205280
6	.187668	.177023	.171808
7	.163544	.153031	.147901
8	.145477	.135043	.129972
9	.131446	.121058	.116029
10	.120241	.109875	.104875
11	.111092	.100730	.095751
12	.103484	.093114	.088149
13	.097062	.086673	.081717
14	.091571	.081156	.076205
15	.086825	.076377	.071428
16	.082685	.072200	.067250
17	.079043	.068516	.063563
18	.075817	.065245	.060288
19	.072940	.062321	.057357
20	.070361	.059691	.054720
21	.068037	.057315	.052335
22	.065932	.055156	.050168
23	.064019	.053188	.048189
24	.062273	.051386	.046376
25	.060674	.049730	.044708
26	.059205	.048203	.043170
27	.057852	.046791	.041745
28	.056603	.045482	.040423
29	.055445	.044264	.039192
30	.054371	.043130	.038044
31	.053372	.042070	.036971
32	.052442	.041078	.035965
33	.051572	.040148	.035020
34	.050760	.039274	.034131
35	.049998	.038451	.033293
36	.049284	.037675	.032502
37	.048613	.036943	.031755
38	.047982	.036250	.031047
39	.047388	.035594	.030375
40	.046827	.034972	.029738
41	.046298	.034382	.029132
42	.045798	.033821	.028555
43	.045325	.033287	.028005
44	.044878	.032778	.027480
45	.044453	.032293	.026979
46	.044051	.031830	.026501
47	.043669	.031388	.026042
48	.043306	.030966	.025603
49	.042962	.030561	.025183
50	.042634	.030174	.024779

TABLE VI.

American Experience.

AGE.	NUMBER LIVING.	NUMBER DYING.	MONETARY VALUES 3½% INTEREST. Annuity.	Single Prem.	Annual Prem.
x	l_x	d_x	a_x	A_x	P_x
20	92,637	723	20.144	.28497	.01348
21	91,914	722	20.013	.28940	.01377
22	91,192	721	19.878	.29399	.01408
23	90,471	720	19.738	.29873	.01441
24	89,751	719	19.592	.30365	.01475
25	89,032	718	19.442	.30873	.01510
26	88,314	718	19.286	.31401	.01548
27	87,596	718	19.124	.31947	.01588
28	86,878	718	18.957	.32512	.01629
29	86,160	719	18.784	.33097	.01673
30	85,441	720	18.605	.33702	.01719
31	84,721	721	18.420	.34328	.01768
32	84,000	723	18.229	.34976	.01819
33	83,277	726	18.030	.35646	.01873
34	82,551	729	17.826	.36339	.01930
35	81,822	732	17.614	.37055	.01991
36	81,090	737	17.395	.37795	.02055
37	80,353	742	17.169	.38560	.02122
38	79,611	749	16.935	.39349	.02194
39	78,862	756	16.695	.40163	.02270
40	78,106	765	16.446	.41003	.02350
41	77,341	774	16.190	.41869	.02436
42	76,567	785	15.926	.42762	.02526
43	75,782	797	15.654	.43681	.02623
44	74,985	812	15.374	.44628	.02726
45	74,173	828	15.087	.45600	.02835
46	73,345	848	14.791	.46600	.02951
47	72,497	870	14.488	.47626	.03075
48	71,627	896	14.177	.48677	.03207
49	70,731	927	13.859	.49752	.03348
50	69,804	962	13.535	.50849	.03499
51	68,842	1001	13.204	.51967	.03659
52	67,841	1044	12.868	.53104	.03829
53	66,797	1091	12.526	.54258	.04011
54	65,706	1143	12.180	.55430	.04206
55	64,563	1199	11.830	.56615	.04413
56	63,364	1260	11.475	.57813	.04634
57	62,104	1325	11.118	.59022	.04871
58	60,779	1394	10.758	.60239	.05123
59	59,385	1468	10.396	.61468	.05394
60	57,917	1546	10.032	.62692	.05683
61	56,371	1628	9.668	.63924	.05992
62	54,743	1713	9.304	.65155	.06323
63	53,030	1800	8.941	.66383	.06678
64	51,230	1889	8.579	.67607	.07058
65	49,341	1980	8.219	.68824	.07465
66	47,361	2070	7.863	.70030	.07902
67	45,291	2158	7.510	.71223	.08370
68	43,133	2243	7.162	.72401	.08871
69	40,890	2321	6.819	.73560	.09408

TABLE VII.

Commutation Columns, American Experience 3½%.

AGE. x	$D_x = v^x l_x$	$N_x = D_{x+1} + D_{x+2} + \&c.$	$M_x = C_x + C_{x+1} + \&c.$	$R_x = M_x + M_{x+1} + \&c.$
20	4655.62	93784.33	1326.75	39728.72
21	4463.08	89321.25	1291.65	38401.97
22	4278.28	85042.97	1257.77	37110.32
23	4100.92	80942.05	1225.09	35852.55
24	3930.71	77011.34	1193.56	34627.46
25	3767.36	73243.98	1163.13	33433.90
26	3610.61	69633.37	1133.77	32270.77
27	3460.15	66173.22	1105.41	31137.00
28	3315.74	62857.48	1078.01	30031.59
29	3177.13	59680.35	1051.53	28953.58
30	3044.08	56636.27	1025.92	27902.05
31	2916.35	53719.92	1001.13	26876.13
32	2793.75	50926.17	977.145	25875.00
33	2676.05	48250.12	953.920	24897.86
34	2563.01	45687.11	931.373	23943.94
35	2454.47	43232.64	909.505	23012.57
36	2350.25	40882.39	888.285	22103.06
37	2250.14	38632.25	867.652	21214.77
38	2153.97	36478.28	847.573	20347.12
39	2061.55	34416.73	827.993	19499.55
40	1972.74	32443.99	808.896	18671.56
41	1887.36	30556.63	790.227	17862.66
42	1805.29	28751.34	771.980	17072.43
43	1726.36	27024.98	754.098	16300.45
44	1650.44	25374.54	736.557	15546.36
45	1577.36	23797.18	719.289	14809.80
46	1507.00	22290.18	702.269	14090.51
47	1439.21	20850.97	685.440	13388.24
48	1373.85	19477.12	668.749	12702.80
49	1310.79	18166.33	652.147	12034.05
50	1249.86	16916.47	635.543	11381.90
51	1190.96	15725.51	618.909	10746.36
52	1133.95	14591.56	602.173	10127.45
53	1078.74	13512.82	585.309	9525.28
54	1025.24	12487.58	568.287	8939.97
55	973.34	11514.24	551.057	8371.68
56	922.96	10591.28	533.592	7820.63
57	874.02	9717.26	515.860	7287.04
58	826.44	8890.82	497.843	6771.18
59	780.18	8110.64	479.529	6273.33
60	735.17	7375.47	460.894	5793.80
61	691.34	6684.13	441.933	5332.91
62	648.68	6035.45	422.643	4890.98
63	607.13	5428.33	403.031	4468.33
64	566.69	4861.64	383.120	4065.30
65	527.33	4334.31	362.931	3682.18
66	489.06	3845.25	342.485	3319.25
67	451.87	3393.39	321.833	2976.77
68	415.78	2977.61	301.030	2654.93
69	380.83	2596.77	280.141	2353.90

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